Performance and scalability of the Block Low-Rank multifrontal factorization

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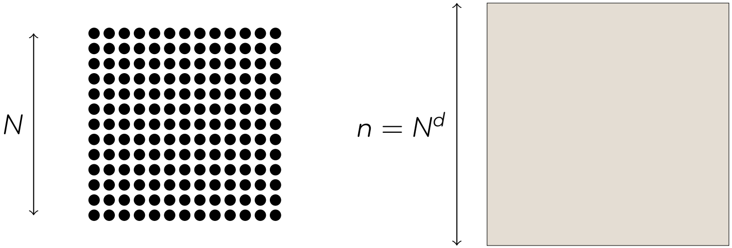
Introduction

Multifrontal (Duff '83) with Nested Dissection (George '73)



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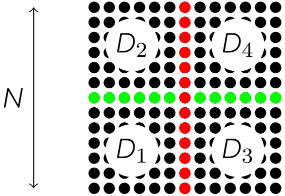
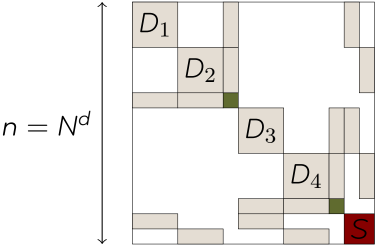
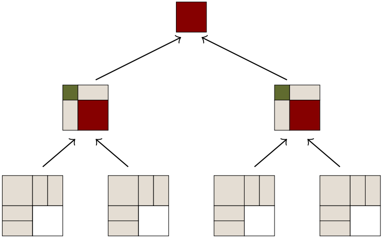


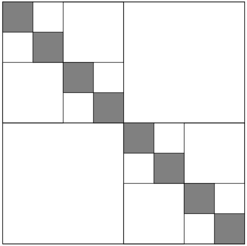




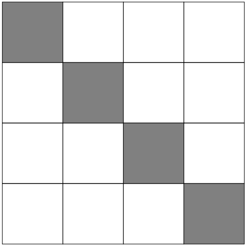
3D problem cost $\propto$

$\rightarrow$ Flops:$O(n^2)$, mem:$O(n^{4/3})$

$\mathcal{H}$ and BLR matrices

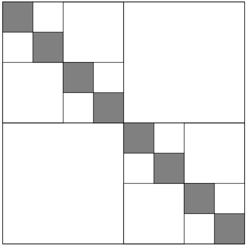


$\mathcal{H}$-matrix



BLR matrix

$\mathcal{H}$ and BLR matrices



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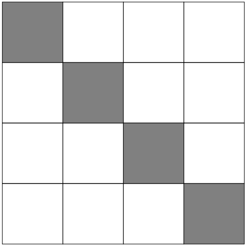
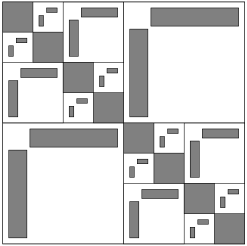


BLR matrix

A block B represents the interaction between two subdomains. If they have a small diameter and are far away their interaction is weak $\Rightarrow$ rank is low.

$\mathcal{H}$ and BLR matrices



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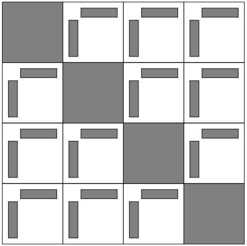

BLR matrix

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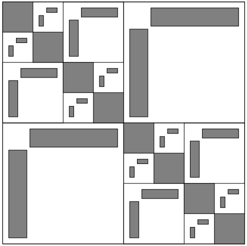
$$\tilde{B} = XY^T \text{ such that } \operatorname{rank}(\tilde{B}) = k_\varepsilon \text{ and } \|B - \tilde{B}\| \leq \varepsilon$$

If $k_\varepsilon \ll \text{size}(B) \Rightarrow$ memory and flops can be reduced with a controlled loss of accuracy ($\leq \varepsilon$)

$\mathcal{H}$ and BLR matrices



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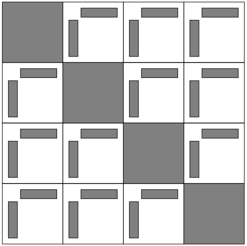
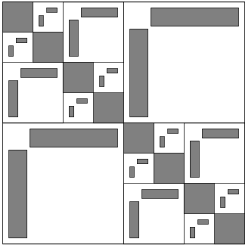


BLR matrix

- Very low theoretical complexity
- Complex, hierarchical structure

- Simple structure
- Theoretical complexity can be as low as the non-fully structured $\mathcal{H}$ case

$\mathcal{H}$ and BLR matrices



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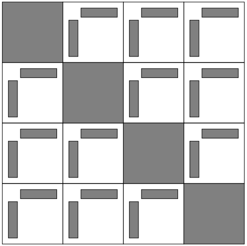

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- Very low theoretical complexity
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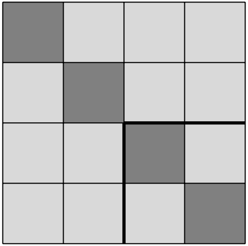
- Simple structure
- Theoretical complexity can be as low as the non-fully structured $\mathcal{H}$ case

$\Rightarrow$ Our hope is to find a good comprise between theoretical complexity and performance/usability

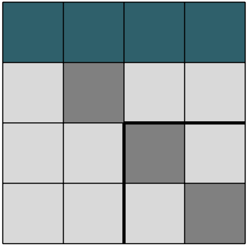
Variants of the BLR factorization

Variants of the BLR LU factorization



- FSCU

Variants of the BLR LU factorization



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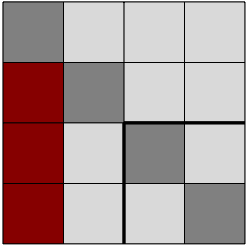

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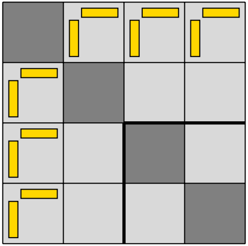

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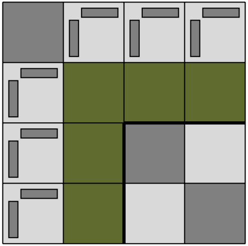

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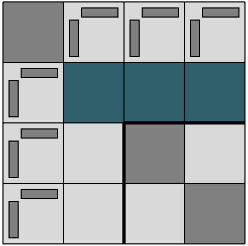

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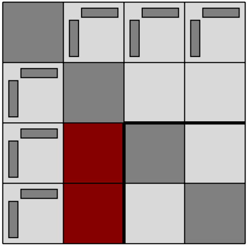

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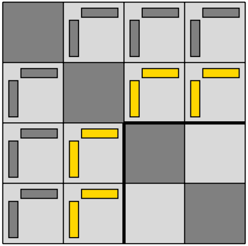

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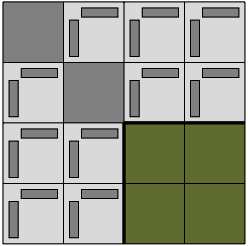

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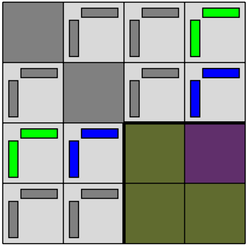

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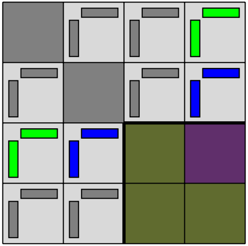

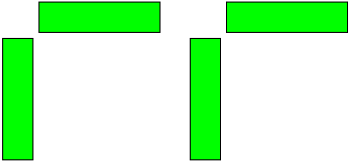

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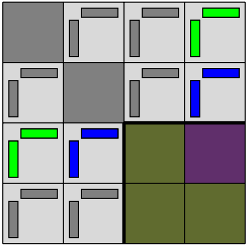

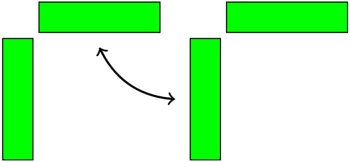

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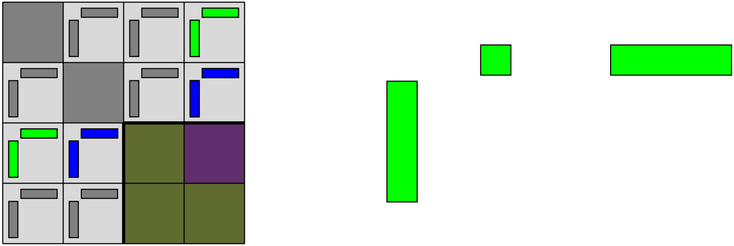

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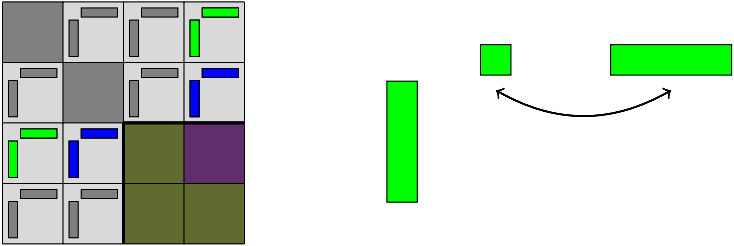

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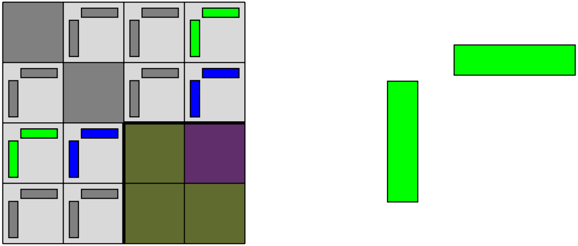

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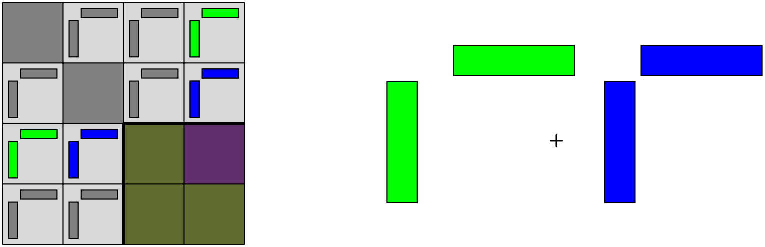

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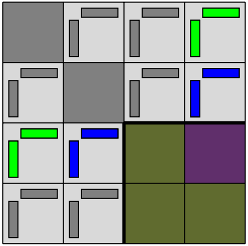

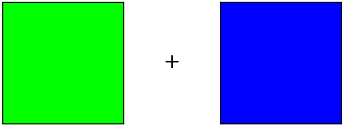

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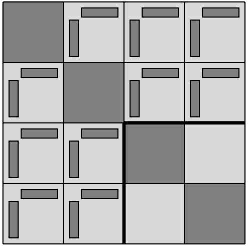

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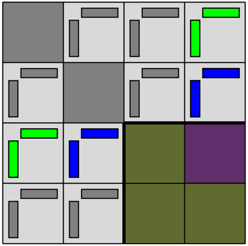

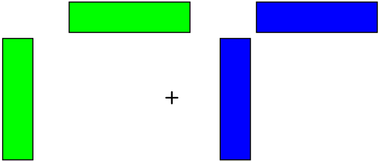

- FSCU (Factor, Solve, Compress, Update)
- FSCU+LUAR

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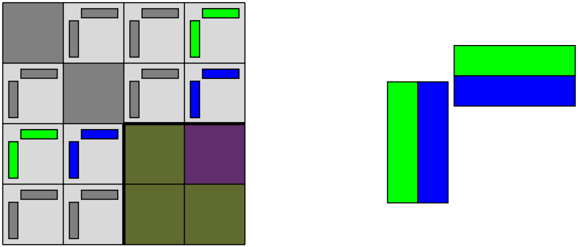
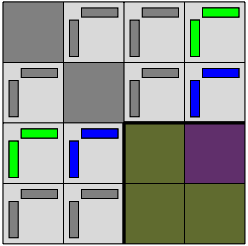
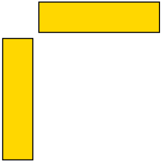


- FSCU (Factor, Solve, Compress, Update)
- FSCU+LUAR
    - Better granularity in Update operations

Variants of the BLR LU factorization





- FSCU (Factor, Solve, Compress, Update)
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  - Better granularity in Update operations
  - Potential for recompression ⇒ flop reduction

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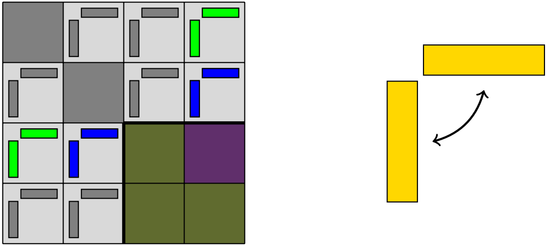

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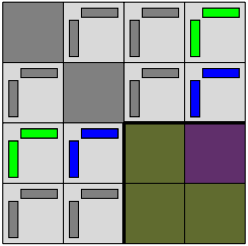

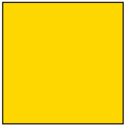

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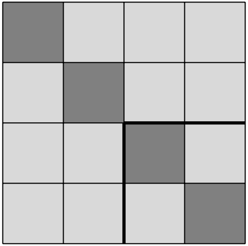

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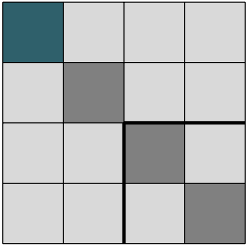

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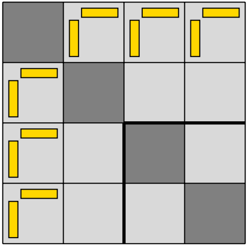

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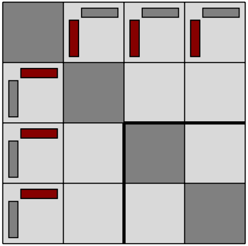

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  - Better BLAS-3/BLAS-2 ratio in Solve operations

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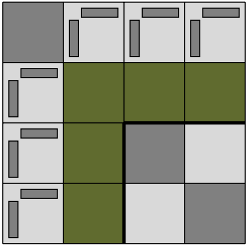

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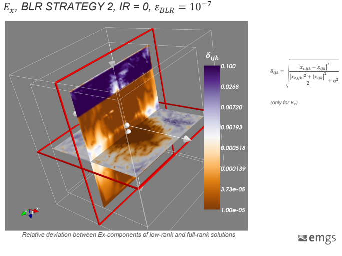
Experimental results

Experimental Setting: Machines

1. Distributed memory experiments are done on the `eos` supercomputer at the CALMIP center of Toulouse (grant 2014-P0989):
   - Two Intel(r) 10-cores Ivy Bridge @ 2,8 GHz
   - Peak per core is 22.4 GF/s
   - 64 GB memory per node
   - Infiniband FDR interconnect

2. Shared memory experiments are done on `grunch` at the LIP laboratory of Lyon:
   - Two Intel(r) 14-cores Haswell @ 2,3 GHz
   - Peak per core is 36.8 GF/s
   - Total memory is 768 GB



3D Electromagnetic Modeling

Maxwell equation

Double complex (z) arithmetic

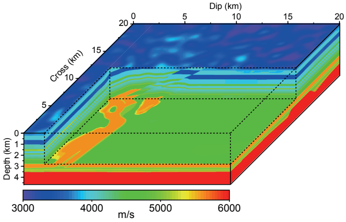
Symmetric LDL^T factorization

Required accuracy: $\varepsilon = 10^{-7}$

Credits: EMGS

matrix	n	nnz	flops	storage
S3	3.3M	43M	78 TF	189 GB
S4	21M	266M	2.5 PF	2.1 TB
D4	30M	384M	3.6 PF	3.0 TB

Full-Rank statistics



3D Seismic Modeling

Helmholtz equation

Single complex (c) arithmetic

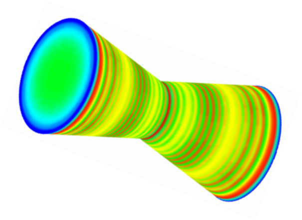
Unsymmetric *LU* factorization

Required accuracy: $\varepsilon = 10^{-3}$

Credits: SEISCOPE

matrix	n	nnz	flops	storage
7Hz	7M	177M	410 TF	211 GB
10Hz	17M	446M	2600 TF	722 GB

Full-Rank statistics



3D Structural Mechanics

Double real (d) arithmetic

Symmetric LDL^T factorization

Required accuracy: $\varepsilon = 10^{-9}$

Credits: Code_Aster (EDF)

matrix	n	nnz	flops	storage
perf008ar	4M	159M	378 TF	148 GB

Full-Rank statistics

Performance on 900 cores

Low-rank threshold ε is set according to the application's target

matrix	MUMPS-(Full-Rank) time	sp-up*	$\%_{peak}$	BLR ε	time
10Hz	1017s	257	26%	10^{-3}	280s
S4	1538s	371	32%	10^{-7}	412s
D4	2221s	373	33%	10^{-7}	515s

*estimated speedup on 90×10 cores

- good speedup and $\%_{peak}$ on 900 cores $\Rightarrow$ good FR reference
- BLR improves performance by a substantial factor of order 4

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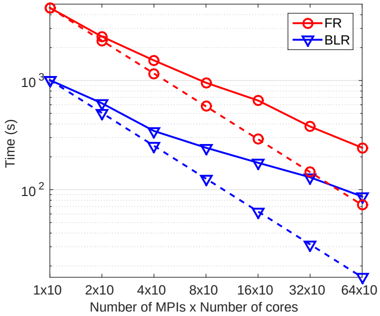
$\Rightarrow$ *but does BLR scale as well as FR?*

Scalability of the BLR factorization (distributed)

MPI+OpenMP parallelism (10 threads/MPI process, 1 MPI/node)

7Hz matrix (extracted from MUMPS-SEISCOPE research work submitted to *Geophysics*)



- each time the number of processes doubles, speedup of ~ 1.6 for FR and ~ 1.5 for BLR

⇒ both FR and BLR scale reasonably well

⇒ ability to maintain gain due to BLR when the number of processes grows

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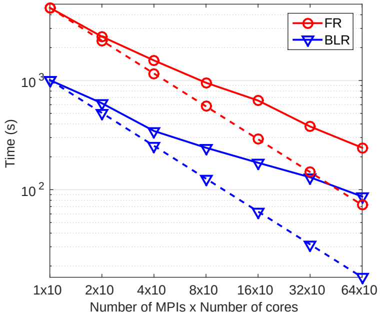
MPI+OpenMP parallelism (10 threads/MPI process, 1 MPI/node)

7Hz matrix (extracted from MUMPS-SEISCOPE research work submitted to *Geophysics*)



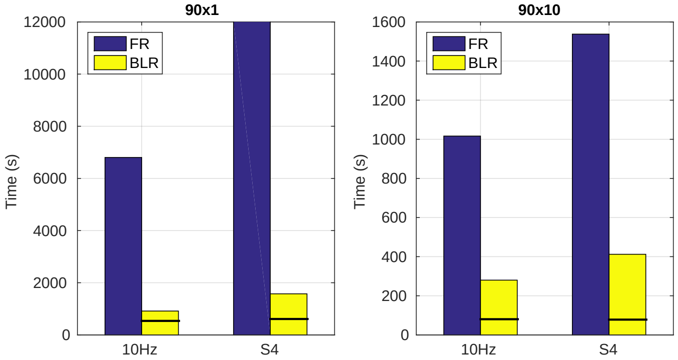
- each time the number of processes doubles, speedup of ~ 1.6 for FR and ~ 1.5 for BLR

$\Rightarrow$ both FR and BLR scale reasonably well

$\Rightarrow$ ability to maintain gain due to BLR when the number of processes grows

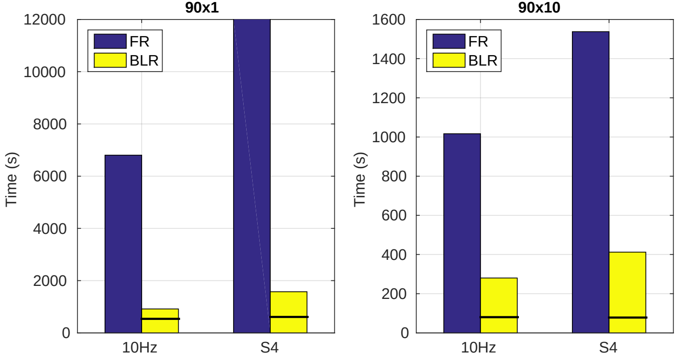
$\Rightarrow$ *so, we are happy?*

Gain due to BLR: impact of multithreading



- gain in flops (black line) does not fully translate into gain in time
- multithreaded efficiency lower in LR than in FR

Gain due to BLR: impact of multithreading



- gain in flops (black line) does not fully translate into gain in time
- multithreaded efficiency lower in LR than in FR

$\Rightarrow$ *improve efficiency of operations and multithreading with variants*

Right Looking Vs. Left-Looking (shared)

Focus on the Update step (which includes the Decompress)

		1 thread		28 threads	
		RL	LL	RL	LL
S3	FR			468s	526s
	BLR	847s	763s	112s	89s
perf008ar	FR			663s	766s
	BLR	2174s	2005s	236s	161s

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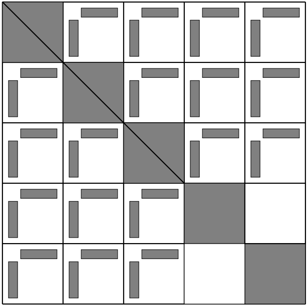

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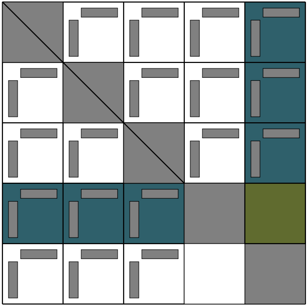

- in RL: FR (green) block is accessed many times; LR (blue) blocks are accessed once
- in LL: FR (green) block is accessed once; LR (blue) blocks are accessed many times

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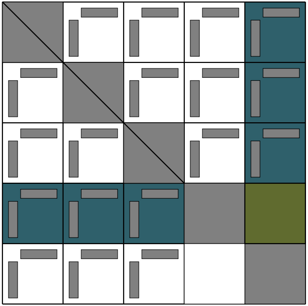

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⇒ lower volume of memory transfers (more critical in multithreaded)

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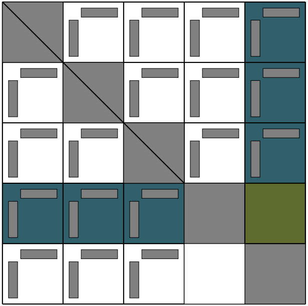

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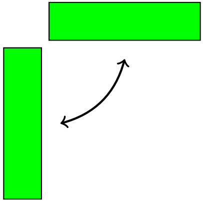
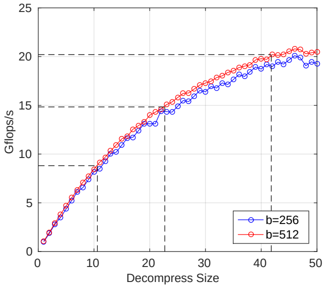
⇒ lower volume of memory transfers (more critical in multithreaded)

⇒ *the Decompress part remains the bottleneck of the Update*

Performance of Update step with LUA(R) (shared, 28 threads)

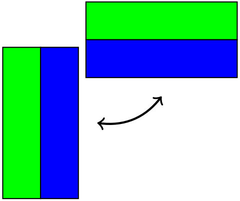


Double precision (d) performance benchmark of Decompress



	S3			perf008ar		
	LL	LUA	LUAR*	LL	LUA	LUAR*
Flops in Update ($\times 10^{12}$)	4.0	4.0	2.9	44	44	33
Avg. decompress size	10.6	41.8	22.7	23.3	89.7	48.1
Time in Update	89s	59s	64s	161s	123s	119s

* All metrics include the Recompression overhead



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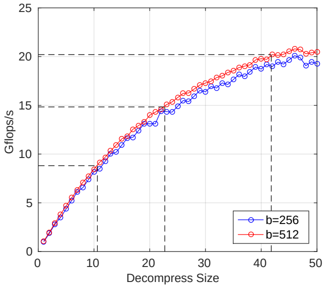

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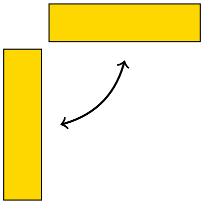
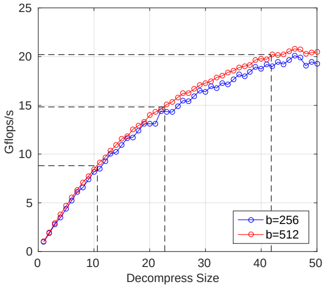


Double precision (d) performance benchmark of Decompress



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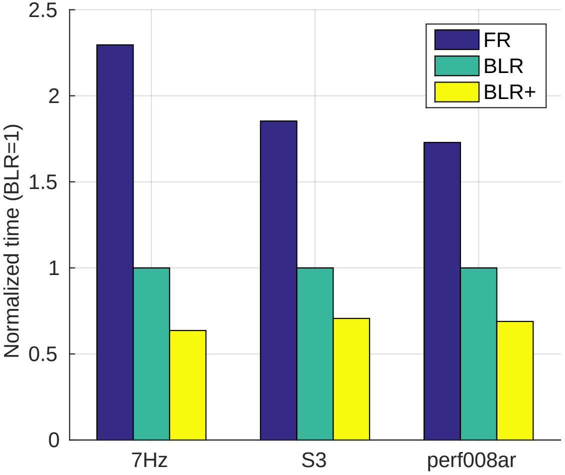
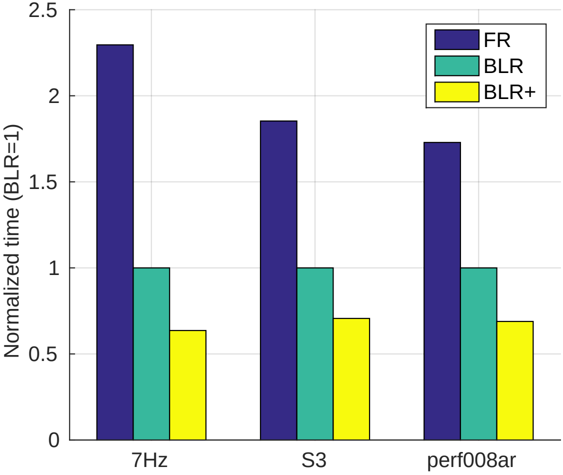
Performance of BLR+ (FCSU+LL+LUA)

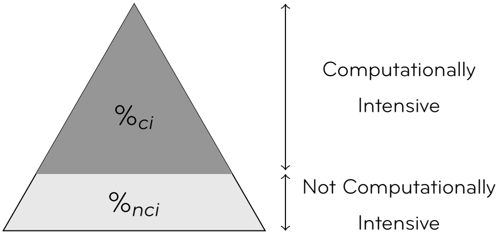


Performance of BLR+ (FCSU+LL+LUA)



$\Rightarrow$ *is there still room for improvement?*

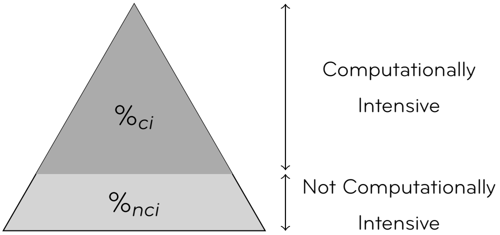
Relative weight of bottom fronts in FR/BLR



	28 threads		
	time	$\%_{nci}$	
FR	585s	18%	

S3 matrix

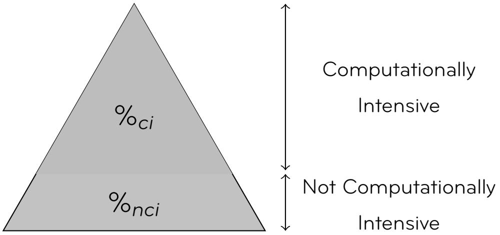
Relative weight of bottom fronts in FR/BLR



	28 threads	
	time	$\%_{nci}$
FR	585s	18%
BLR	315s	34%

S3 matrix

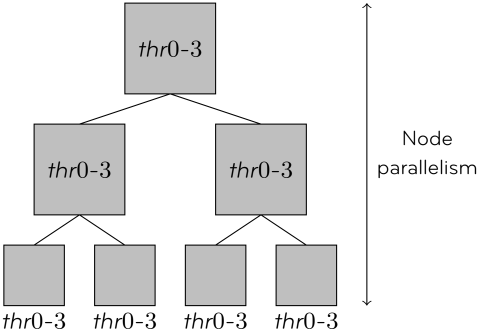
Relative weight of bottom fronts in FR/BLR



	28 threads	
	time	$\%_{nci}$
FR	585s	18%
BLR	315s	34%
BLR+	223s	48%

S3 matrix

Exploiting tree-based multithreading in MF solvers



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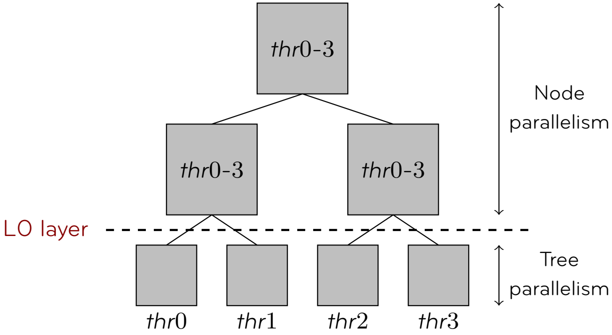

- Work based on W. M. Sid-Lakhdar's PhD thesis
  - L0 layer computed with a variant of the Geist-Ng algorithm
  - NUMA-aware implementation
  - use of Idle Core Recycling technique (variant of work-stealing)

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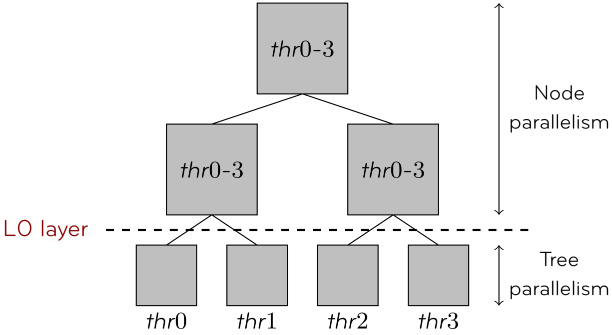

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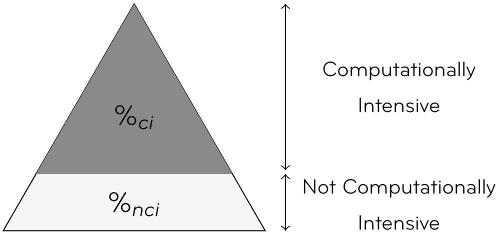
$\Rightarrow$ *how big an impact can tree-based multithreading make?*

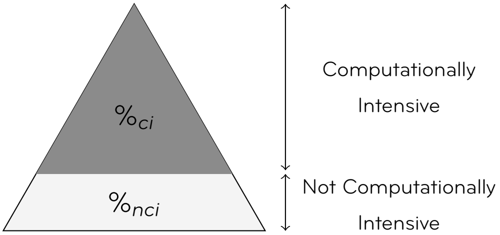
Impact of tree-based multithreading on BLR/BLR+



	28 threads		28 threads + tree MT	
	time	$\%_{nci}$	time	$\%_{nci}$
FR	585s	18%	519s	8%
BLR	315s	34%		
BLR+	223s	48%		

S3 matrix

Impact of tree-based multithreading on BLR/BLR+



	28 threads		28 threads + tree MT	
	time	$\%_{nci}$	time	$\%_{nci}$
FR	585s	18%	519s	8%
BLR	315s	34%	239s	10%
BLR+	223s	48%	136s	9%

S3 matrix

Conclusion and perspectives

Performance results on real-life problems

- Standard BLR variant (FSCU) achieves speedups of order 4 on 900 cores w.r.t. FR
- Scalability of BLR factorization is comparable to FR one
- But flop reduction is not fully translated into performance gain, especially with multithreading
- Improved BLR variants (BLR+) possess better properties (efficiency, granularity, volume of communications, number of operations)
- Tree-based multithreading becomes critical in BLR, especially BLR+
- Combination of tree MT and BLR+ leads to speedups of order 3 on 28 threads w.r.t. standard BLR

Perspectives

- Implementation and performance analysis of the BLR variants in distributed memory (MPI+OpenMP parallelism)
- Efficient strategies to recompress LR updates
- Pivoting strategies compatible with the BLR variants
- Influence of the BLR variants on the accuracy of the factorization

Acknowledgements

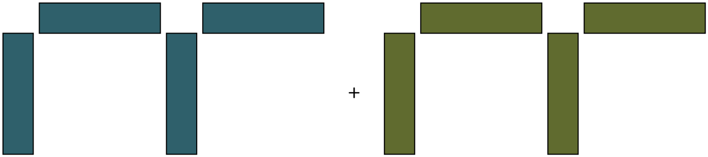
- CALMIP and LIP for providing access to the machines
- EMGS, SEISCOPE and EDF for providing the test matrices
- LSTC members for scientific discussions

Thanks!

Questions?

Backup Slides

Accumulator recompression



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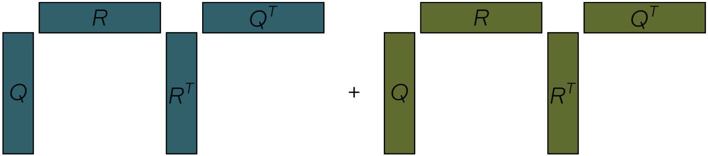

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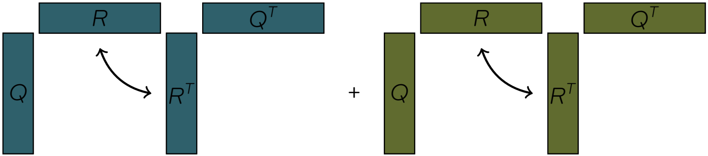

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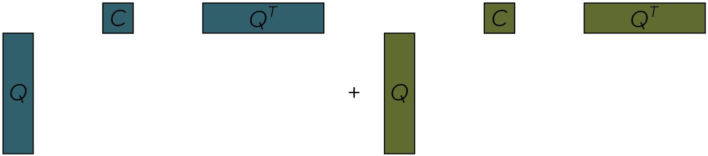

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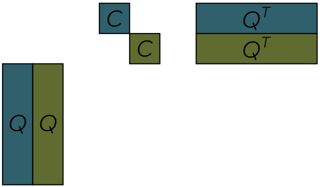

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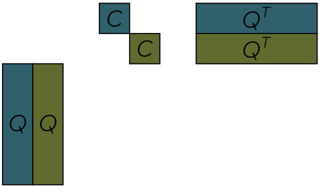

- Weight recompression on $\{C_i\}_i$

  $\Rightarrow$ With absolute threshold ε, each C_i can be compressed separately

- Redundancy recompression on $\{Q_i\}_i$

  $\Rightarrow$ Bigger recompression overhead, when is it worth it?