Pseudozero Set of Interval Polynomials

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The 21st Annual ACM Symposium on Applied Computing
Dijon, France, April 23-27, 2006
Outline of the talk

I — Pseudozero set
• Definition and computation

II — Pseudozero set of interval polynomials
• Real pseudozero set of polynomials
• Presentation of PSIP
Pseudozeros: definition, computation and motivation
Pseudozero set: definition

Perturbation:
Neighborhood of polynomial $p$

$$N_\varepsilon(p) = \{ \hat{p} \in \mathbb{C}_n[z] : \| p - \hat{p} \| \leq \varepsilon \}.$$ 

Definition of the $\varepsilon$-pseudozero set:

$$Z_\varepsilon(p) = \{ z \in \mathbb{C} : \hat{p}(z) = 0 \text{ for } \hat{p} \in N_\varepsilon(p) \}.$$ 

$\| \cdot \|$ a norm on the vector of the coefficients of $p$

This set is formed by the zeros of polynomials “near $p$”.

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Pseudozeros : brief survey of existing references

► Mosier (1986) : Definition and study form the $\infty$-norm.
► Hinrichsen and Kelb : spectral value sets
► Trefethen and Toh (1994) : Study for the 2-norm.
   pseudozeros $\approx$ pseudospectra of the companion matrix.
► Chatelin and Frayssé (1996) : propose a Synthesis in Lectures on Finite Precision Computations (SIAM)
► Zhang (2001) : Study of the influence of the basis for the 2-norm (condition number of the evaluation).
► Karow (2003) : thesis on Spectral value sets
Pseudozeros are easily computable

**Theorem:**
The \( \varepsilon \)-pseudozeros set satisfies

\[
Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|z\|_*} \leq \varepsilon \right\},
\]

where \( z = (1, z, \ldots, z^n) \) and \( \| \cdot \|_* \) is the dual norm of \( \| \cdot \| \),

\[
\|y\|_* = \sup_{x \neq 0} \frac{|y^*x|}{\|x\|}
\]
Algorithm of computation

Algorithm to draw the \( \varepsilon \)-pseudzero set:

1. We mesh a square containing all the roots of \( p \) (MATLAB command: `meshgrid`).
2. We compute \( g(z) := \frac{|p(z)|}{\|z\|_*} \) for all the nodes \( z \) in the grid.
3. We draw the contour level \( |g(z)| = \varepsilon \) (MATLAB command: `contour`).
A famous example

Pseudozero set of the \textit{Wilkinson} polynomial

\[ W_{20} = (z - 1)(z - 2) \cdots (z - 20), \]
\[ = z^{20} - 210z^{19} + \cdots + 20!. \]

We perturb only the coefficient of \( z^{19} \) with \( \varepsilon = 2^{-23} \).

One use the weighted-norm \( \| \cdot \|_\infty \):

\[ \| p \|_\infty = \max_i \left| \frac{p_i}{m_i} \right| \text{ with } m_i \text{ non negative} \]

with \( m_{19} = 1 \), \( m_i = 0 \) otherwise and the convention \( m/0 = \infty \) if \( m > 0 \) and \( 0/0 = 0 \).
Pseudozero set of interval polynomials
Interval polynomial

An interval polynomial is a polynomial whose coefficients are real intervals.

We denote by $\mathbb{IR}[z]$ the set of interval polynomials and by $\mathbb{IR}_n[z]$ the set of interval polynomials with degree at most $n$.

Let $p \in \mathbb{IR}_n[z]$. We can write

$$p(z) = \sum_{i=0}^{n} [a_i, b_i] z^i.$$ 

The zeros of an interval polynomial is the set

$$Z(p) := \{z \in \mathbb{C} : \text{there exist } m_i \in [a_i, b_i], i = 0 : n \text{ such that } \sum_{i=0}^{n} m_i z^i = 0 \}.$$ 

$\implies$ Compute $Z(p)$. 

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Definition of real pseudozero set

Let \( p = \sum_{i=0}^{n} p_i z^i \) be a polynomial of \( \mathbb{R}_n[z] \)

Perturbations:
Real neighborhood of \( p \)

\[
N^R_\varepsilon(p) = \{ \hat{p} \in \mathbb{R}_n[z] : \|p - \hat{p}\| \leq \varepsilon \}.
\]

Definition of the real \( \varepsilon \)-pseudozero set

\[
Z^R_\varepsilon(p) = \{ z \in \mathbb{C} : \hat{p}(z) = 0 \text{ for } \hat{p} \in N^R_\varepsilon(p) \}.
\]
Computation of the real pseudozero set

**Theorem:**
The real $\varepsilon$-pseudozero set satisfies

$$Z_{\varepsilon}^R(p) = Z(p) \cup \left\{ z \in \mathbb{C} \setminus Z(p) : h(z) := d(G_R(z), \mathbb{R}G_I(z)) \geq \frac{1}{\varepsilon} \right\},$$

where $d$ is defined for $x, y \in \mathbb{R}^{n+1}$ by

$$d(x, \mathbb{R}y) = \inf_{\alpha \in \mathbb{R}} \|x - \alpha y\|_*$$

and where $G_R(z)$, $G_I(z)$ are the real and imaginary part of

$$G(z) = \frac{1}{p(z)}(1, z, \ldots, z^n)^T, \ z \in \mathbb{C} \setminus Z(p)$$

Can be viewed as a special case of *spectral value set* [Karow 03]
What for $\mathbb{R} \cap Z^R_\epsilon(p)$?

**Lemma.** Given $z \in \mathbb{R}$, $z$ belongs to $Z^R_\epsilon(p)$ if and only if $z$ belongs to $Z_\epsilon(p)$.

Draw the complex pseudozero set or the real pseudozero set on the real axis is similar.
Some properties

The function $d$ defined for $x, y \in \mathbb{R}^{n+1}$ by

$$d(x, Ry) = \inf_{\alpha \in \mathbb{R}} \|x - \alpha y\|_*$$

satisfies

$$d(x, Ry) = \begin{cases} 
\sqrt{\|x\|_2^2 - \frac{(x,y)^2}{\|y\|_2^2}} & \text{if } y \neq 0, \\
\|x\|_2 & \text{if } y = 0
\end{cases}$$

for the norm $\| \cdot \|_2$

$$d(x, Ry) = \begin{cases} 
\min_{i=0:n; y_i \neq 0} \|x - (x_i/y_i)y\|_1 & \text{if } y \neq 0, \\
\|x\|_1 & \text{if } y = 0
\end{cases}$$

for the norm $\| \cdot \|_\infty$
Some properties (cont’d)

**Proposition 1:**
The real $\varepsilon$-pseudozero set $Z^R_\varepsilon(p)$ is symmetric with respect to the real axis.

**Proposition 2:**
The real $\varepsilon$-pseudozero set $Z^R_\varepsilon(p)$ is included in the complex $\varepsilon$-pseudozero set.
Algorithm to draw real pseudozero set

Drawing of real $\varepsilon$-pseudozero set :

1. We mesh a square containing all the roots of $p$ (MATLAB command : meshgrid).
2. We compute $h(z) := d(G_R(z), R G_I(z))$ for all the nodes $z$ in the grid.
3. We draw the contour level $|h(z)| = \frac{1}{\varepsilon}$ (MATLAB command : contour).
Pseudozero set with weighted norm

\[ p(z) = \sum_{i=0}^{n} p_i z^i. \]

- Identification of \( p \) with the vector \((p_0, p_1, \ldots, p_n)^T\)
- \( d := (d_0, \ldots, d_n)^T \in \mathbb{R}^{n+1} \) represents the weight of the coefficients of \( p \)
- \( \| \cdot \|_{\infty,d} \) defined by

\[ \|p\|_{\infty,d} = \max_{i=0:n} \{|p_i|/|d_i|\}. \]
Let us denote $p_c$ the central polynomial defined by

$$p_c(z) = \sum_{i=0}^{n} c_i z^i,$$

with $c_i = (a_i + b_i)/2$.

Let us denote $d_i := (b_i - a_i)/2$.

**Proposition :**
With the notation above, we have

$$\mathbf{Z}(p) = Z_{\varepsilon}^R(p_c) \text{ with } \varepsilon = 1.$$
Example 1

\[ p(z) = [1, 2]z^4 + [3, 3.2]z^3 + [10, 14]z^2 + [3, 5\sqrt{2}]z + [5, 7] \]
Example 2

\[ p(z) = z^3 + z^2 + [3, 8]z + [1.5, 4] \]
Problem : choice of the grid

Lemma:
Let \( p(z) = \sum_{i=0}^{n} [a_i, b_i] z^i \) an interval polynomial and

\[
R := 1 + \frac{\max_{i=0:n} \{\max \{|a_i|, |b_i|\}\}}{\min \{|a_n|, |b_n|\}}.
\]

Then

\[ Z(p) \subset B(O, R), \]

where \( B(O, R) \) the ball in \( \mathbb{C} \) of centre \( O \) and radius \( R \).
Problems : discontinuities

Lemma [Hinrichsen et Kelb] :

The function

\[ d : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}_+, \quad (x, y) \mapsto d(x, R_y) \]

is continue for all \((x, y)\) with \(y \neq 0\) or \(x = 0\) and discontinue for all \((x, 0) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, x \neq 0\).

Those discontinuities imply some difficulties for drawing near the real axis.

Solution : on the real axis, we draw complex pseudozero set.
Presentation of PSIP

A tool to draw zeros of interval polynomials
Presentation of PSIP (cont’d)

- a graphical interface for MATLAB (version 6.5 (R13))
- computation of grid that contains all the zeros
- possibilities of zoom and mesh refinement

Limitations:
- problem if the leading interval contains 0
- problems with discontinuities
Conclusion and future work

We have presented
  • a tool to draw pseudozero set of interval polynomial

Future work
  • certification of the drawing