Pseudozero Set of Multivariate Polynomials

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General motivations

- Polynomials appear in almost all areas in scientific computing and engineering
- The relationships between industrial applications and polynomial systems solving studied by the European Community Project FRISCO
- Applications in Computer Aided Design and Modeling, Mechanical Systems Design, Signal Processing and Filter Design, Civil Engineering, Robotics, Simulation
- The wide range of use of polynomial systems needs to have fast and reliable methods to solve them
  - symbolic approach based either on the theory of Gröbner basis or on the theory of resultants
  - numeric approach based on iterative methods like Newton's method or homotopy continuation methods
  - recently, hybrid methods, combining both symbolic and numeric methods

Dealing with uncertainties

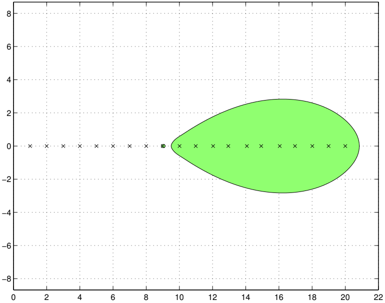
- In practice, from situations arising in science or engineering, the data are known only to a limited accuracy
- Analytical sensitivity analysis introduces a condition number that bounds the magnitudes of the (first order) changes of the roots with respect to the coefficient perturbations
- Continuous sensitivity analysis, introduced by Ostrowski, considers the uncertainty of the coefficients as a continuity problem. The most powerful tool of this last type of methods seems to be the pseudozero set of a polynomial

An example for the univariate case

Computing the zeros of the Wilkinson polynomial of degree 20

$$\begin{aligned} W(x) &= (x-1)(x-2)\cdots(x-20) \\ &= x^{20} - 210x^{19} + \cdots + 20! \end{aligned}$$

Uncertainty of 2^{-23} on the coefficient of x^{19}



Brief survey of existing references

- Mosier (1986): Definition and study form the ∞-norm.
- Trefethen and Toh (1994): Study for the 2-norm.
  pseudozeros $\approx$ pseudospectra of the companion matrix.
- Chatelin and Frayssé (1996): propose a Synthesis in *Lectures on Finite Precision Computations* (SIAM)
- Stetter (1999,2004): *Numerical polynomial algebra*. Generalization of the previous works.
- Zhang (2001): Study of the influence of the basis for the 2-norm (condition number of the evaluation).
- Hoffman, Madden, Zhang (2003): the multivariate case
- Corless, Kai, Watt (2003): algorithms for the multivariate case

Definitions (1/3)

A monomial in the n variables $z_1, \ldots, z_n$ is the power product

$$z^j := z_1^{j_1} \cdots z_n^{j_n}, \quad \text{with } j = (j_1, \ldots, j_n) \in \mathbb{N}^n;$$

j is the exponent and $|j| := \sum_{\sigma=1}^{n} j_\sigma$ the *degree* of the monomial z^j.

Definition 1

A complex (real) polynomial in n variables is a finite linear combination of monomials in n variables with coefficients from $\mathbb{C}$ (from $\mathbb{R}$),

$$p(z) = p(z_1, \ldots, z_n) = \sum_{(j_1,\ldots,j_n) \in J}^{n} a_{j_1 \cdots j_n} z_1^{j_1} \cdots z_n^{j_n} = \sum_{j \in J} a_j z^j.$$

$\mathcal{P}^n(\mathbb{C})$ ($\mathcal{P}^n(\mathbb{R})$) represents the set of all complex (real) polynomials in n variables.

Definitions (2/3)

Given $p = \sum_{j \in J} a_j z^j \in \mathcal{P}^n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$
$\longrightarrow$ $|J|$ the number of elements of J

If $|J| = M$ and let $\|\cdot\|$ be a norm on $\mathbb{K}^M$
$\longrightarrow$ $\|p\|$ is the norm of the vector $a = (\ldots, a_j, \ldots, j \in J)$

Given a norm $\|\cdot\|$ on $\mathbb{K}^N$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the dual norm is defined by $\|x\|_* := \sup_{\|y\|=1} |y^T x|$.

Given a vector $x \in \mathbb{K}^N$, there exists a dual vector $y \in \mathbb{K}^N$ with $\|y\| = 1$ satisfying $x^T y = \|x\|_*$.

Norms	Dual norms
$\|x\|_1 := \sum_j \|x_j\|$	$\|x\|_1^* = \max_j \|x_j\| = \|x\|_\infty$
$\|x\|_2 := (\sum_j \|x_j\|^2)^{1/2}$	$\|x\|_2^* = (\sum_j \|x_j\|^2)^{1/2} = \|x\|_2$
$\|x\|_\infty := \max_j \|x_j\|$	$\|x\|_\infty^* = \sum_j \|x_j\| = \|x\|_1$

Definitions (3/3)

Given $\varepsilon > 0$, the ε-neighborhood $N_\varepsilon(p)$ of the polynomial $p \in \mathcal{P}^n(\mathbb{K})$ is the set of all polynomials of $\mathcal{P}^n(\mathbb{K})$ with $\widetilde{p} = \sum_{j \in \widetilde{J}} \widetilde{a}_j z^j \in \mathcal{P}^n(\mathbb{K})$ with support $\widetilde{J} \subset J$ and $\|\widetilde{p} - p\| \leq \varepsilon$.

Definition 2

A value $z \in \mathbb{K}^n$ is an ε-pseudozero of a polynomial $p \in \mathcal{P}^n$ if it is a zero of some polynomial $\widetilde{p}$ in $N_\varepsilon(p)$.

Definition 3

The ε-pseudozero set of a polynomial $p \in \mathcal{P}^n$ (denoted by $Z_\varepsilon(p)$) is the set of all the ε-pseudozeros,

$$Z_\varepsilon(p) := \{z \in \mathbb{K}^n : \exists \widetilde{p} \in N_\varepsilon(p), \quad \widetilde{p}(z) = 0\}.$$

Pseudozeros of complex multivariate polynomials (1/2)

Theorem 1 (Stetter)

The complex ε-pseudozero set of $p = \sum_{j \in J} a_j z^j \in \mathcal{P}^n(\mathbb{C})$ verifies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C}^n : g(z) := \frac{|p(z)|}{\|\mathbf{z}\|_*} \leq \varepsilon \right\}$$

where $\mathbf{z} := (\ldots, |z|^j, \ldots, j \in J)^T$.

Pseudozeros of complex multivariate polynomials (2/2)

Corollary 1 (Stetter)

The complex ε-pseudozero set of $P = \{p_1, \dots, p_k\}$, $k \in \mathbb{N}$ verifies

$$Z_\varepsilon(P) = \left\{ z \in \mathbb{C}^n : \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \leq \varepsilon \text{ for } l = 1, \dots, k \right\},$$

where $\mathbf{z_l} := (\dots, |z|^j, \dots, j \in J_l)^T$.

We restrict our attention to situations where P as well as all the systems in $N_\varepsilon(P)$ are 0-dimensional, that is, if the solution of the system is non-empty and finite.

Theorem 2 (Stetter)

Each system $\widetilde{P} \in N_\varepsilon(P)$ has the same number of zeros (counting multiplicities) in a fixed pseudozero set connected component of $Z_\varepsilon(P)$.

Pseudozeros of real multivariate polynomials: definition

A real ε-neighborhood of p is the set of all polynomials of $\mathcal{P}^n(\mathbb{R})$, close enough to p, that is to say,

$$N_\varepsilon^R(p) = \{\widetilde{p} \in \mathcal{P}^n(\mathbb{R}) : \|p - \widetilde{p}\| \leq \varepsilon\}.$$

The real ε-pseudozero set of p is defined to include all the zeros of the real ε-neighborhood of p :

$$Z_\varepsilon^R(p) = \left\{z \in \mathbb{C}^n : \widetilde{p}(z) = 0 \text{ for } \widetilde{p} \in N_\varepsilon^R(p)\right\}.$$

For $\varepsilon = 0$, the pseudozero set $Z_0^R(p)$ is the set of the roots of p we denote $Z(p)$.

Pseudozeros of real multivariate polynomials: computation

Distance of a point $x \in \mathbb{R}^N$ from the linear subspace $\mathbb{R}y = \{\alpha y, \alpha \in \mathbb{R}\}$

$$d(x, \mathbb{R}y) = \inf_{\alpha \in \mathbb{R}} \|x - \alpha y\|_*,$$

Theorem 3

The real ε-pseudozero set of $p = \sum_{j \in J} a_j z^j \in \mathcal{P}^n(\mathbb{R})$ verifies

$$Z_\varepsilon^R(p) = Z(p) \cup \left\{ z \in \mathbb{C}^n \backslash Z(p) : h(z) := d(G_R(z), \mathbb{R}G_I(z)) \geq \frac{1}{\varepsilon} \right\},$$

where $G_R(z)$ and $G_I(z)$ are the real and imaginary parts of

$$G(z) = \frac{1}{p(z)}(\ldots, z^j, \ldots, j \in J)^T, \; z \in \mathbb{C}^n \backslash Z(p).$$

Computing the distance

- computing real ε-pseudozero set $Z_\varepsilon^R(p)$ needs to evaluate the distance $d(G_R(z), \mathbb{R}G_I(z))$.
- the 2-norm $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$ the corresponding inner product

$$d(x, \mathbb{R}y) = \begin{cases} \sqrt{\|x\|_2^2 - \frac{\langle x,y\rangle^2}{\|y\|_2^2}} & \text{if } y \neq 0, \\ \|x\|_2 & \text{if } y = 0. \end{cases}$$

- the ∞-norm,

$$d(x, \mathbb{R}y) = \begin{cases} \min_{\substack{i=0:n \\ y_i \neq 0}} \|x - (x_i/y_i)y\|_1 & \text{if } y \neq 0, \\ \|x\|_1 & \text{if } y = 0. \end{cases}$$

- other p-norm with $p \neq 2, \infty$, no easy computable formula to calculate $d(x, \mathbb{R}y)$.

Real pseudozeros of polynomial systems

Corollary 2

The real ε-pseudozero set of $P = \{p_1, \dots, p_k\}$, $k \in \mathbb{N}$ verifies

$$Z_\varepsilon^R(P) = \bigcap_{l=1}^{k} \left(Z(p_l) \cup \left\{ z \in \mathbb{C}^n \backslash Z(p_l) : d(G_R^l(z), \mathbb{R}G_I^l(z)) \geq \frac{1}{\varepsilon} \right\} \right)$$

where $G_R^l(z)$ and $G_I^l(z)$ are the real and imaginary parts of

$$G^l(z) = \frac{1}{p_l(z)} (\dots, z^j, \dots, j \in J_l)^T, \; z \in \mathbb{C}^n \backslash Z(p_l).$$

Visualization of pseudozero sets (1/5)

- The descriptions of $Z_\varepsilon(P)$ and $Z_\varepsilon^R(P)$ given previously make it possible to compute, plot and visualize pseudozero set of multivariate polynomials.
- The pseudozero set is a subset of $\mathbb{C}^n$ which can only be seen by its projections on low dimensional spaces that is often $\mathbb{C}$.

We have written a MATLAB program to compute and visualize these projections. This program requires the Symbolic Math Toolbox.

Visualization of pseudozero sets (2/5)

For a given $v \in \mathbb{C}^n$, let $Z_\varepsilon(P,j,v)$ be the projection of $Z_\varepsilon(P)$ onto the z_j-space around v. Then, it follows that for $P = \{p_1, \dots, p_k\}$,

$$Z_\varepsilon(P,j,v) = \left\{ z \in \mathbb{C}^n : z_i = v_i,\ i \neq j,\ \max_{l=1,\dots,k} \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \leq \varepsilon \right\},$$

where $\mathbf{z_l} := (\dots, |z|^j, \dots, j \in J_l)^T$.

One way for visualizing $Z_\varepsilon(P,j,v)$ is to plot the values of the projection of

$$\mathrm{ps}(z) := \log_{10}\left(\max_{l=1,\dots,k} \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \right)$$

over a set of grid points around v in z_j-space.

Visualization of pseudozero sets (3/5)

In the same way, we define for a given $v \in \mathbb{C}^n$, $Z_\varepsilon^R(P, j, v)$ by the projection of $Z_\varepsilon^R(P)$ onto the z_j-space around v. It follows that for $P = \{p_1, \dots, p_k\}$,

$$Z_\varepsilon^R(P, j, v) = \left\{ z \in \mathbb{C}^n : z_i = v_i,\ i \neq j,\ \max_{l=1,\dots,k} d(G_R^l(z), \mathbb{R}G_I^l(z))^{-1} \leq \varepsilon \right\}$$

where $G_R^l(z)$ and $G_I^l(z)$ are the real and imaginary parts of

$$G^l(z) = \frac{1}{p_l(z)}(\dots, z^j, \dots, j \in J_l)^T,\ z \in \mathbb{C}^n \backslash Z(p_l).$$

One way for visualizing $Z_\varepsilon^R(P, j, v)$ is still to plot the values of the projection of

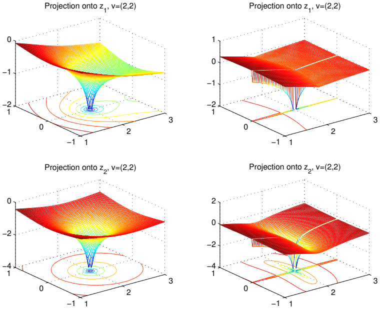
$$\mathrm{ps}^R(z) := \log_{10} \left(\max_{l=1,\dots,k} d(G_R^l(z), \mathbb{R}G_I^l(z))^{-1} \right)$$

over a set of grid points around v in z_j-space.

Visualization of pseudozero sets (4/5)

We examine the following system using the 2-norm: two unit balls intersection at $(2,2)$,

$$P_1 = \begin{cases} p_1 = (z_1-1)^2 + (z_2-2)^2 - 1, \\ p_2 = (z_1-3)^2 + (z_2-2)^2 - 1. \end{cases}$$



Projections of the complex pseudozero set (on the left) and the real pseudozero set (on the right) of P_1

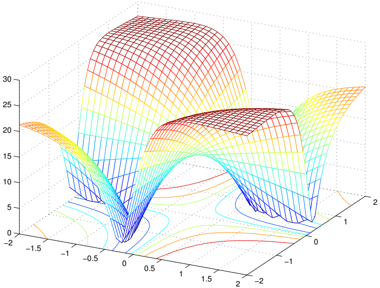
We can be only interested in the real zeros of a polynomial systems. In this case, we can only draw $\mathbb{R}^n \cap Z_\varepsilon^R(P)$.

$$P_2 = \begin{cases} p_1 = z_1^2 + z_2^2 - 1, \\ p_2 = 25 z_1 z_2 - 12. \end{cases}$$

We have computed the function

$$g(x,y) = \max_{l=1,2} \frac{p_l(x,y)}{\|\mathbf{z_l}\|_*},$$

with $\mathbf{z_l} := (\ldots, |x+iy|^j, \ldots, j \in J_l)^T$.



Projection of the real pseudozero set of P_2

Conclusion and future work

- Approximate polynomials are unavoidable in numerous application fields and in finite precision environment.
- Plotting pseudozero set can give qualitative and sometimes quantitative interesting informations about the behavior of these approximate polynomials.

We hope that pseudozero set will be used as much as pseudospectra.

Thank you for your attention

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