Computation of pseudozero abscissa

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Motivations

Polynomial coefficients are often approximate values

Three well known sources of approximation are considered in scientific computation :

(1) errors due to discretization and truncation,

(2) errors due to roundoff, and

(3) errors due to uncertainty in the data.

$\Longrightarrow$ Use tools designed for such approximate polynomials in control theory

Outline of the talk

1 — Pseudozero set

- Definition
- Computation

2 — Applications of pseudozeros in control theory

- Robust stability of polynomials
- Pseudozero abscissa of polynomials

Pseudozeros :
definition, computation

Pseudozero set : definition

Let p be a given polynomial of $\mathbf{C}_n[z]$

Perturbation :

Neighborhood of polynomial p

$$N_\varepsilon(p) = \{\widehat{p} \in \mathbf{C}_n[z] : \|p - \widehat{p}\| \leqslant \varepsilon\} .$$

Definition of the ε-pseudozero set :

$$Z_\varepsilon(p) = \{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\} .$$

$\|\cdot\|$ a norm on the vector of the coefficients of p

Pseudozero set : the set of the zeros of polynomials "near p".

Pseudozeros are easily computable

Theorem [Stetter] :
The ε-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|\underline{z}\|_*} \leqslant \varepsilon \right\},$$

where $\underline{z} = (1, z, \ldots, z^n)$ and $\| \cdot \|_*$ is the dual norm of $\| \cdot \|$,

$$\|y\|_* = \sup_{x \neq 0} \frac{|y^* x|}{\|x\|}$$

Pseudozero set : algorithm of computation

1. We mesh a square containing all the roots of p (MATLAB command : `meshgrid`).
2. We compute $g(z) := \frac{|p(z)|}{\|\underline{z}\|_*}$ for all the nodes z of the grid.
3. We plot the contour level $|g(z)| = \varepsilon$ (MATLAB command : `contour`).

Initialization :

- Find a square containing all the roots of p and all the pseudozeros.
- Find a grid step that separates all the roots.

A famous example

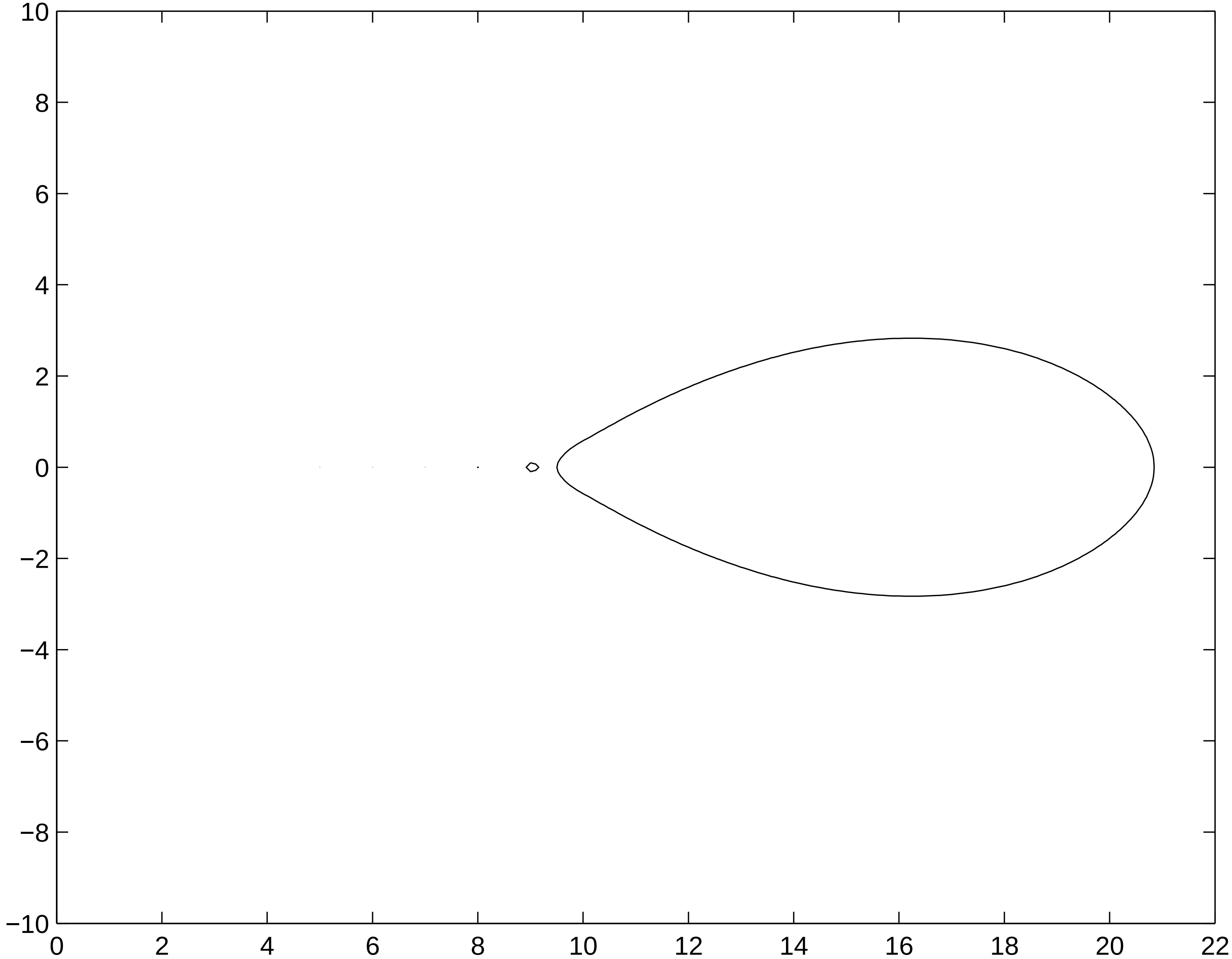
Pseudozero set of the *Wilkinson* polynomial

$$\begin{aligned} W_{20} &= (z-1)(z-2)\cdots(z-20), \\ &= z^{20} - 210z^{19} + \cdots + 20!. \end{aligned}$$

We only perturb the coefficient of z^{19} with $\varepsilon = 2^{-23}$.
One uses the weighted-norm $\|\cdot\|_\infty$:

$$\|p\|_\infty = \max_i \frac{|p_i|}{m_i} \text{ with } m_i \text{ non negative}$$

with $m_{19} = 1$, $m_i = 0$ otherwise and the convention $m/0 = \infty$ if $m > 0$ and $0/0 = 0$.



10
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16
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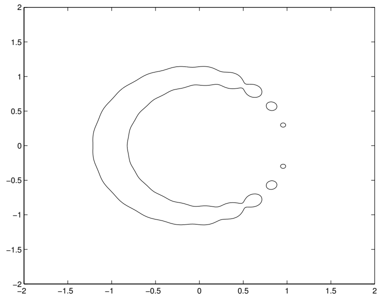
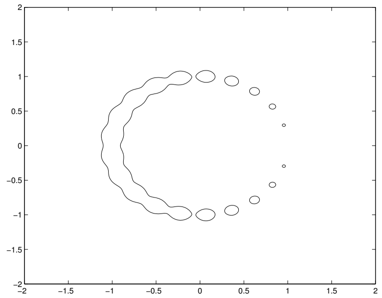
20
22

Evolution of ε-pseudozero w.r.t ε

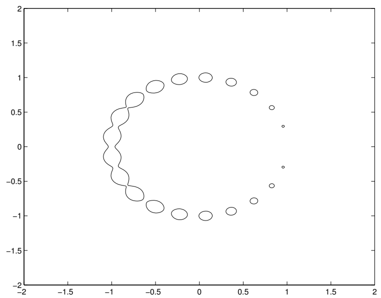
Pseudozero set of the polynomial $p(z) = 1 + z + \cdots + z^{20}$ for different values of ε (for the 2-norm).



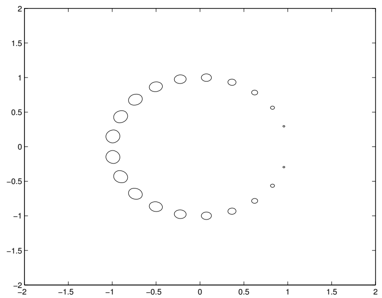
(a) $\varepsilon = 10^{-1}$



(b) $\varepsilon = 10^{-1.2}$



(c) $\varepsilon = 10^{-1.3}$



(d) $\varepsilon = 10^{-1.4}$

Pseudozeros : brief survey of existing references

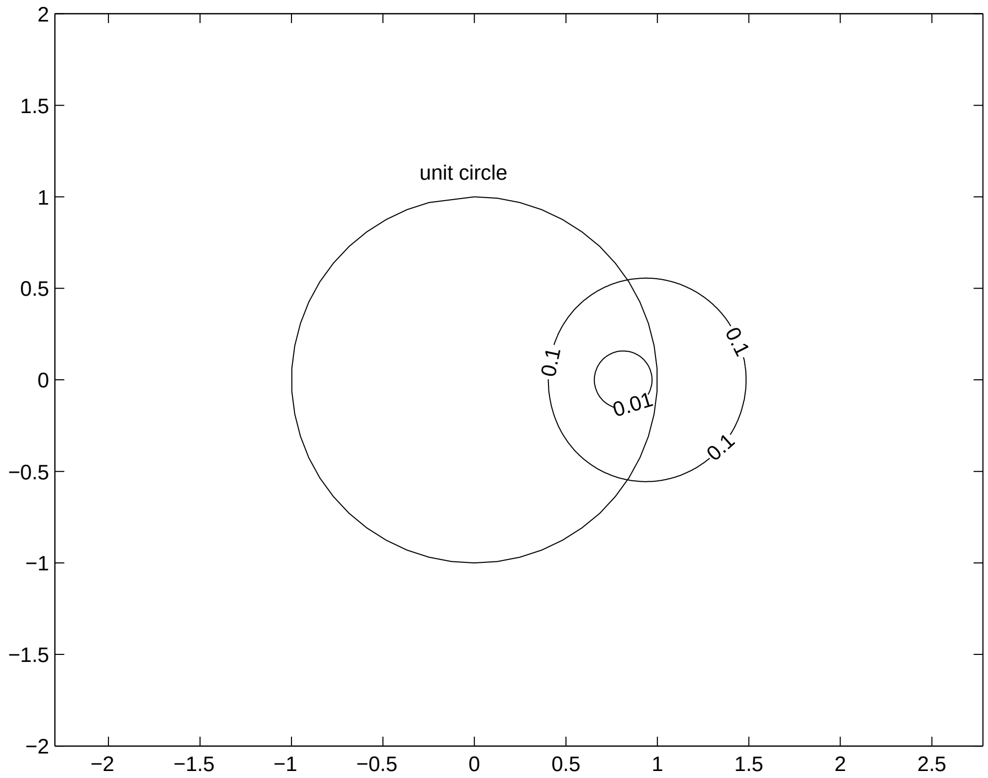
- Mosier (1986) : Definition and study for the ∞-norm.
- Hinrichsen and Kelb (1993) : Spectral value sets.
- Trefethen and Toh (1994) : Study for the 2-norm.
  pseudozeros $\approx$ pseudospectra of the companion matrix.
- Zhang (2001) : Study the influence of the basis for the 2-norm (condition number of the evaluation).
- Stetter (2004) : *Numerical Polynomial Algebra* (SIAM). Generalization of the previous works.

Robust stability
and
Pseudozero abscissa

Schur robust stability in control theory

Schur stability : $|\text{roots of } p| < 1$.

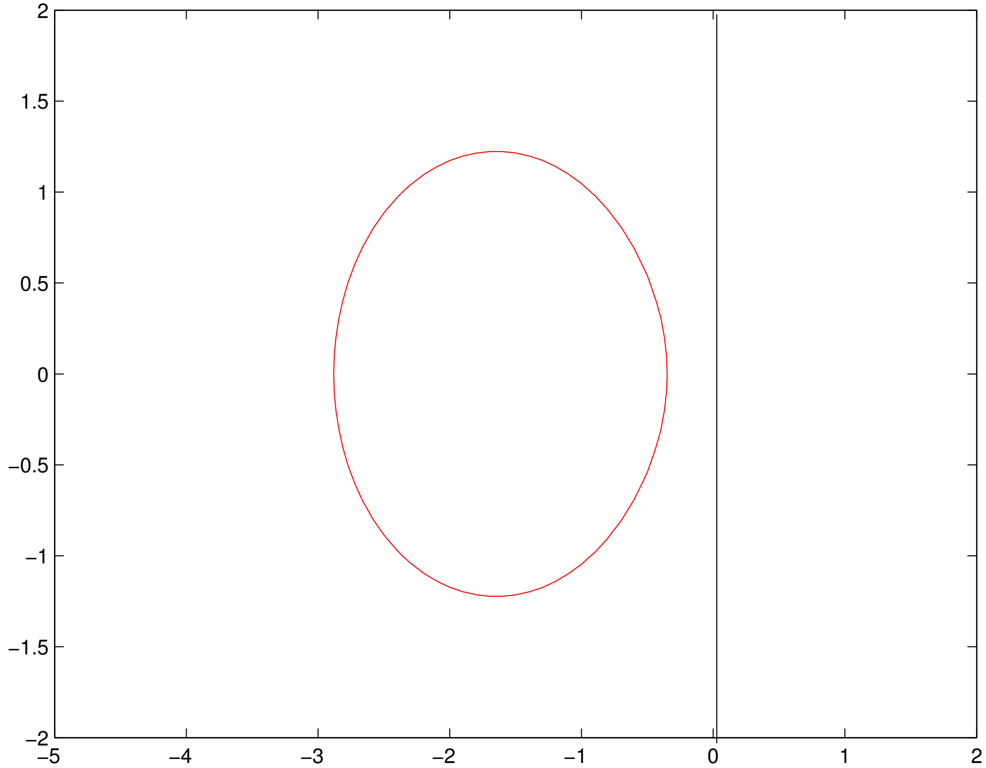
ε-pseudozero set of $p(z) = (z - 0.8)^2$ for $\varepsilon = 0.1$ and $\varepsilon = 0.01$.



Hurwitz robust stability in control theory

Hurwitz stability : Real part of roots of $p < 0$.

ε-pseudozero set of $p(z) = (z+1)^2$ for $\varepsilon = 0.4$.



Computation of pseudozero abscissa

$\mathcal{P}_n$: polynomials of $\mathbf{C}[X]$ of degree at most n
$\mathcal{M}_n$: monic polynomials of $\mathcal{P}_n$ of degree n
$\|\cdot\|$: the 2-norm of the coefficients of a polynomial

Definition. *A polynomial is* stable *if all its roots have negative real part and* unstable *otherwise (Hurwitz stability).*

The function *abscissa* $a : \mathcal{P} \to \mathbf{R}$ is defined by

$$a(p) = \max\{\mathrm{Re}(z) : p(z) = 0\}.$$

A polynomial p is stable $\Longleftrightarrow a(p) < 0$

Motivation

In control theory, transfer functions are often written as $H(p) = \frac{N(p)}{D(p)}$ where N and D are polynomials.

The system is stable if D is a stable polynomial .

Question : if D is stable, is it still stable when perturbed

(we assume that D is monic)

Pseudozero abscissa mapping

ε-pseudozero abscissa mapping $a_\varepsilon : \mathcal{P}_n \to \mathbf{R}$:

$$a_\varepsilon(p) = \max\{\mathrm{Re}(z) : z \in Z_\varepsilon(p)\}.$$

Statement of the problem :

Given a polynomial $p \in \mathcal{M}_n$, let us compute $a_\varepsilon(p)$.

A polynomial p is ε-robustly stable $\iff a_\varepsilon(p) < 0$

Our solutions

Tools

- an explicit formula that defines the pseudozeros
- the continuous dependency of the roots w.r.t the polynomial coefficients
- Sturm sequences to count the real roots
- *criss-cross* algorithm

The results : 3 algorithms

- a plotting algorithm
- a bisection algorithm
- a criss-cross algorithm

Pseudozero set for monic polynomials

Perturbation : Neighborhood of polynomial p

$$N_\varepsilon(p) = \{\widehat{p} \in \mathcal{M}_n : \|p - \widehat{p}\| \leqslant \varepsilon\}\,.$$

Definition of the ε-pseudozero set :

$$Z_\varepsilon(p) = \{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\}\,.$$

$\|\cdot\|$ is the 2-norm on the vector of the coefficients of p

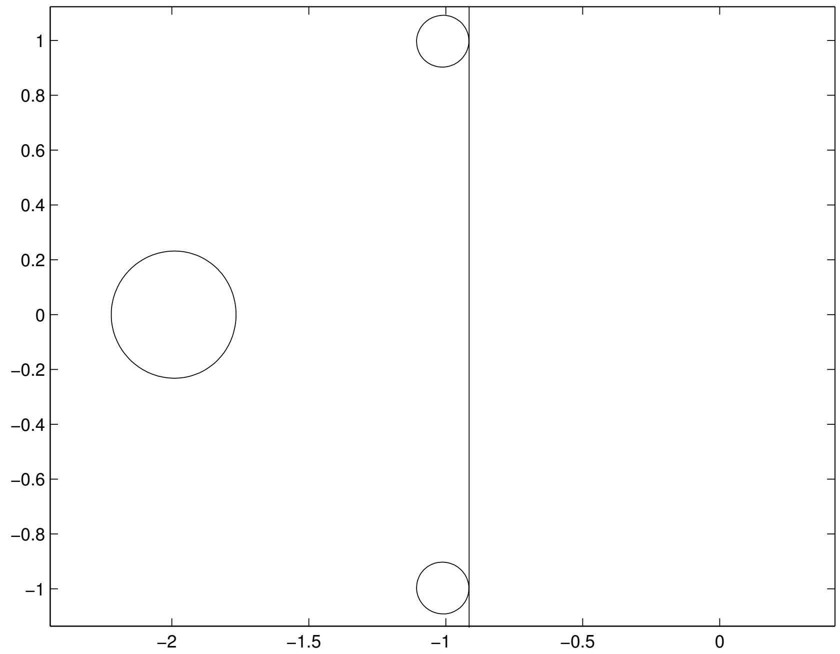
The ε-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|\underline{z}\|} \leqslant \varepsilon \right\},$$

where $\underline{z} = (1, z, \ldots, z^{n-1})$

A plotting algorithm

- Draw the ε-pseudozero set
- Draw the vertine line that intersects the right-most point within the ε-pseudozero set



ε-pseudozero set of $p(z) = z^3 + 4z^2 + 6z + 4$ for $\varepsilon = 0.1$

$$a_\varepsilon(p) \approx -0.9$$

Another characterization of $Z_\varepsilon(p)$

Let us denote $h_{p,\varepsilon} : \mathbf{R}^2 \to \mathbf{R}$, the function

$$h_{p,\varepsilon}(x,y) = |p(x+iy)|^2 - \varepsilon^2 \sum_{j=0}^{n-1} (x^2+y^2)^j.$$

Then one has

$$Z_\varepsilon(p) = \{(x,y) \in \mathbf{R}^2 : h_{p,\varepsilon}(x,y) \leqslant 0\}$$

$\Longrightarrow$ $h_{p,\varepsilon}(\cdot, y)$ et $h_{p,\varepsilon}(x, \cdot)$ are polynomials of degree $2n$.

Theorem. *For any real $x \geqslant a(p)$, the equation $h_{p,\varepsilon}(x,y) = 0$ has a real solution y if and only if $x \leqslant a_\varepsilon(p)$.*

A bisection algorithm

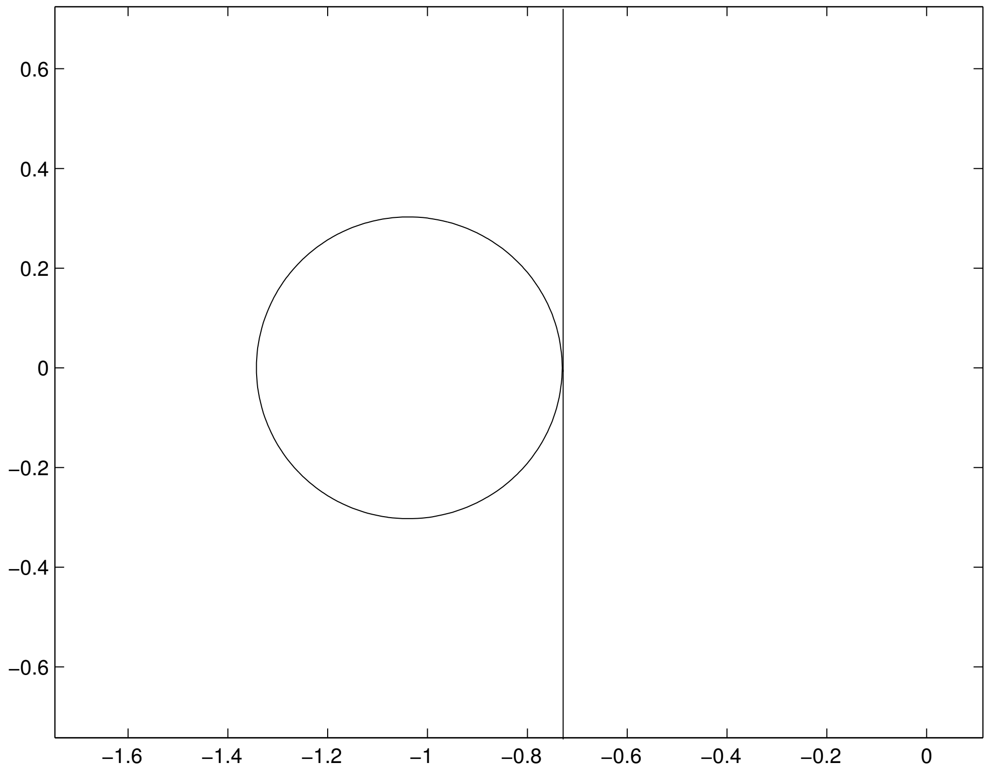
Require : a stable polynomial p, the parameter ε and a tolerance τ

Ensure : a number α such that $|\alpha - a_\varepsilon(p)| \leqslant \tau$

$\gamma := a(p), \quad \delta := 1 + \|p\| + \varepsilon$
while $|\gamma - \delta| > \tau$ **do**
    $x := \frac{\gamma+\delta}{2}$
    if the equation $h_{p,\varepsilon}(x, y) = 0$ has a real solution **then**
        $\delta := x$
    else
        $\gamma := x$
    end if
end while
return $\alpha = \frac{\gamma+\delta}{2}$

Numerical simulation

For $p(z) = z^5 + 5^4 + 10z^3 10z^2 + 5z + 1$, the algorithm gives $a_\varepsilon(p) \approx -0.719669$ for $\varepsilon = 0.001$ and $\tau = 0.00001$



ε-pseudozero set of $p(z) = z^5 + 5^4 + 10z^3 10z^2 + 5z + 1$ for $\varepsilon = 0.001$

A criss-cross algorithm

Require : a polynomial p, the parameter ε

1: Initialize : $x^1 = a(p)$ and $r = 1$

2: Vertical search : find open intervals $I_1^r, \dots, I_{l_r}^r$ where $h(x^r, y) < 0$ for $y \in \cup_{k=1}^{l_r} I_k^r$

3: Horizontal search : for each I_k^r, define $\omega_k^r = \text{midpoint}(I_k^r)$ and find the largest real zeros x_k^r of the function $h(\cdot, \omega_k^r)$ for $k = 1 : l_r$

4: Define $x^{r+1} = \max\{x_k^r, k = 1, \dots, l_r\}$, increment r by one and return to Step 2.

Conclusion and future work

Conclusion :

Pseudozero set provides

- a better understanding of the effect of coefficient perturbations
- some applications for robust stability

Future work :

- an analysis of the convergence of the criss-cross algorithm (we hope a quadratic convergence)
- an implementation of the criss-cross algorithm
- a generalization to pseudospectra of polynomial matrices