Computation of pseudozero abscissa

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Motivations

Polynomial coefficients are often approximate values

Three well known sources of approximation are considered in scientific computation:

(1) errors due to discretization and truncation,
(2) errors due to roundoff, and
(3) errors due to uncertainty in the data.

Use tools designed for such approximate polynomials in control theory
Outline of the talk

1 — Pseudozero set
   • Definition
   • Computation

2 — Applications of pseudozeros in control theory
   • Robust stability of polynomials
   • Pseudozero abscissa of polynomials
Pseudozeros: definition, computation
Pseudozero set: definition

Let \( p \) be a given polynomial of \( \mathbb{C}_n[z] \)

**Perturbation:**
 Neighborhood of polynomial \( p \)

\[
N_{\varepsilon}(p) = \{ \hat{p} \in \mathbb{C}_n[z] : \| p - \hat{p} \| \leq \varepsilon \}.
\]

**Definition of the \( \varepsilon \)-pseudozero set:**

\[
Z_{\varepsilon}(p) = \{ z \in \mathbb{C} : \hat{p}(z) = 0 \text{ for } \hat{p} \in N_{\varepsilon}(p) \}.
\]

\( \| \cdot \| \) a norm on the vector of the coefficients of \( p \)

Pseudozero set: the set of the zeros of polynomials “near \( p \)”.

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Pseudozeros are easily computable

**Theorem [Stetter]**:
The $\varepsilon$-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|z\|_*} \leq \varepsilon \right\},$$

where $z = (1, z, \ldots, z^n)$ and $\| \cdot \|_*$ is the dual norm of $\| \cdot \|$, 

$$\|y\|_* = \sup_{x \neq 0} \frac{|y^*x|}{\|x\|}$$
Pseudozero set : algorithm of computation

1. We mesh a square containing all the roots of $p$ (MATLAB command : meshgrid).
2. We compute $g(z) := \frac{|p(z)|}{\|z\|_*}$ for all the nodes $z$ of the grid.
3. We plot the contour level $|g(z)| = \varepsilon$ (MATLAB command : contour).

Initialization :
- Find a square containing all the roots of $p$ and all the pseudozeros.
- Find a grid step that separates all the roots.
A famous example

Pseudozero set of the *Wilkinson* polynomial

\[
W_{20} = (z - 1)(z - 2) \cdots (z - 20), \\
= z^{20} - 210z^{19} + \cdots + 20!.
\]

We only perturb the coefficient of \(z^{19}\) with \(\varepsilon = 2^{-23}\).
One uses the weighted-norm \(\| \cdot \|_\infty\):

\[
\|p\|_\infty = \max_i \frac{|p_i|}{m_i} \text{ with } m_i \text{ non negative}
\]

with \(m_{19} = 1, m_i = 0\) otherwise and the convention \(m/0 = \infty\) if \(m > 0\) and \(0/0 = 0\).
Evolution of $\varepsilon$-pseudozero w.r.t $\varepsilon$

Pseudozero set of the polynomial $p(z) = 1 + z + \cdots + z^{20}$ for different values of $\varepsilon$ (for the 2-norm).

(a) $\varepsilon = 10^{-1}$

(b) $\varepsilon = 10^{-1.2}$

(c) $\varepsilon = 10^{-1.3}$

(d) $\varepsilon = 10^{-1.4}$
Pseudozeros : brief survey of existing references

- Mosier (1986) : Definition and study for the $\infty$-norm.
- Trefethen and Toh (1994) : Study for the 2-norm.
  - pseudozeros $\approx$ pseudospectra of the companion matrix.
- Zhang (2001) : Study the influence of the basis for the 2-norm (condition number of the evaluation).
Robust stability
and
Pseudozero abscissa
Schur robust stability in control theory

**Schur stability**: \(|\text{roots of } p| < 1\).

\(\varepsilon\)-pseudozero set of \(p(z) = (z - 0.8)^2\) for \(\varepsilon = 0.1\) and \(\varepsilon = 0.01\).
Hurwitz robust stability in control theory

Hurwitz stability: Real part of roots of $p < 0$.

$\varepsilon$-pseudozero set of $p(z) = (z + 1)^2$ for $\varepsilon = 0.4$. 

![Diagram](image-url)
Computation of pseudozero abscissa

\( \mathcal{P}_n \): polynomials of \( \mathbb{C}[X] \) of degree at most \( n \)

\( \mathcal{M}_n \): monic polynomials of \( \mathcal{P}_n \) of degree \( n \)

\( \| \cdot \| \): the 2-norm of the coefficients of a polynomial

**Definition.** A polynomial is **stable** if all its roots have negative real part and unstable otherwise (Hurwitz stability).

The function abscissa \( a : \mathcal{P} \rightarrow \mathbb{R} \) is defined by

\[
a(p) = \max \{ \text{Re}(z) : p(z) = 0 \}.
\]

A polynomial \( p \) is stable \( \iff \) \( a(p) < 0 \)
In control theory, transfer functions are often written as \( H(p) = \frac{N(p)}{D(p)} \) where \( N \) and \( D \) are polynomials.

The system is stable if \( D \) is a stable polynomial.

Question: if \( D \) is stable, is it still stable when perturbed?

(we assume that \( D \) is monic)
Pseudozero abscissa mapping

$\varepsilon$-pseudozero abscissa mapping $a_\varepsilon : \mathcal{P}_n \rightarrow \mathbb{R}$:

$$a_\varepsilon(p) = \max\{\text{Re}(z) : z \in Z_\varepsilon(p)\}.$$ 

**Statement of the problem:**

Given a polynomial $p \in \mathcal{M}_n$, let us compute $a_\varepsilon(p)$.

A polynomial $p$ is $\varepsilon$-robustly stable $\iff a_\varepsilon(p) < 0$.
Our solutions

Tools
- an explicit formula that defines the pseudozeros
- the continuous dependency of the roots w.r.t the polynomial coefficients
- Sturm sequences to count the real roots
- criss-cross algorithm

The results: 3 algorithms
- a plotting algorithm
- a bisection algorithm
- a criss-cross algorithm
Pseudozero set for monic polynomials

Perturbation: Neighborhood of polynomial \( p \)

\[
N_\varepsilon(p) = \{ \hat{p} \in \mathcal{M}_n : \| p - \hat{p} \| \leq \varepsilon \} .
\]

Definition of the \( \varepsilon \)-pseudozero set:

\[
Z_\varepsilon(p) = \{ z \in \mathbb{C} : \hat{p}(z) = 0 \text{ for } \hat{p} \in N_\varepsilon(p) \} .
\]

\( \| \cdot \| \) is the 2-norm on the vector of the coefficients of \( p \)

The \( \varepsilon \)-pseudozeros set satisfies

\[
Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|z\|} \leq \varepsilon \right\} ,
\]

where \( z = (1, z, \ldots, z^{n-1}) \)
A plotting algorithm

- Draw the $\varepsilon$-pseudozero set
- Draw the vertine line that intersects the right-most point within the $\varepsilon$-pseudozero set

$\varepsilon$-pseudozero set of $p(z) = z^3 + 4z^2 + 6z + 4$ for $\varepsilon = 0.1$

$a_{\varepsilon}(p) \approx -0.9$
Another characterization of \( Z_\varepsilon(p) \)

Let us denote \( h_{p,\varepsilon} : \mathbb{R}^2 \to \mathbb{R} \), the function

\[
h_{p,\varepsilon}(x, y) = |p(x + iy)|^2 - \varepsilon^2 \sum_{j=0}^{n-1} (x^2 + y^2)^j.
\]

Then one has

\[
Z_\varepsilon(p) = \{(x, y) \in \mathbb{R}^2 : h_{p,\varepsilon}(x, y) \leq 0\}
\]

\( \implies h_{p,\varepsilon}(\cdot, y) \) et \( h_{p,\varepsilon}(x, \cdot) \) are polynomials of degree \( 2n \).

**Theorem.** For any real \( x \geq a(p) \), the equation \( h_{p,\varepsilon}(x, y) = 0 \) has a real solution \( y \) if and only if \( x \leq a_\varepsilon(p) \).
A bisection algorithm

Require: a stable polynomial $p$, the parameter $\varepsilon$ and a tolerance $\tau$
Ensure: a number $\alpha$ such that $|\alpha - a_\varepsilon(p)| \leq \tau$

1: $\gamma := a(p)$, $\delta := 1 + \|p\| + \varepsilon$
2: while $|\gamma - \delta| > \tau$ do
3: $x := \frac{\gamma + \delta}{2}$
4: if the equation $h_{p,\varepsilon}(x, y) = 0$ has a real solution then
5: $\delta := x$
6: else
7: $\gamma := x$
8: end if
9: end while
10: return $\alpha = \frac{\gamma + \delta}{2}$
Numerical simulation

For $p(z) = z^5 + 5^4 + 10z^310z^2 + 5z + 1$, the algorithm gives $a_\varepsilon(p) \approx -0.719669$ for $\varepsilon = 0.001$ and $\tau = 0.00001$

$\varepsilon$-pseudozero set of $p(z) = z^5 + 5^4 + 10z^310z^2 + 5z + 1$ for $\varepsilon = 0.001$
A criss-cross algorithm

Require : a polynomial $p$, the parameter $\varepsilon$

1: Initialize : $x^1 = a(p)$ and $r = 1$

2: Vertical search : find open intervals $I^r_1, \ldots, I^r_{l_r}$ where $h(x^r, y) < 0$ for $y \in \bigcup_{k=1}^{l_r} I^r_k$

3: Horizontal search : for each $I^r_k$, define $\omega^r_k = \text{midpoint}(I^r_k)$ and find the largest real zeros $x^r_k$ of the function $h(\cdot, \omega^r_k)$ for $k = 1 : l_r$

4: Define $x^{r+1} = \max\{x^r_k, k = 1, \ldots, l_r\}$, increment $r$ by one and return to Step 2.
Conclusion and future work

Conclusion :
Pseudozero set provides
• a better understanding of the effect of coefficient perturbations
• some applications for robust stability

Future work :
• an analysis of the convergence of the criss-cross algorithm (we hope a quadratic convergence)
• an implementation of the criss-cross algorithm
• a generalization to pseudospectra of polynomial matrices