A comparison of real and complex pseudozero sets for polynomials with real coefficients

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RNC'6, 6th Conference on Real Numbers and Computers
Schloss Dagstuhl, Germany, November 15–17, 2004

Motivations

Polynomial coefficients are often approximate values

Three well known sources of approximation in scientific computation :

(1) errors due to discretization and truncation,

(2) errors due to roundoff,

(3) errors due to uncertainty in the data.

$\Longrightarrow$ Use tools designed for such approximate polynomials

Existing tools

- condition number : bound the magnitude of change of the roots with respect to the coefficient perturbations (Gautschi, Wilkinson)
- interval arithmetic : yield over-set that enclose the perturbed roots (Moore)
- stochastic arithmetic : use if the coefficient uncertainty obeys a given probabilistic behavior (Vigne, Chesneaux)
- pseudozero set : the set of roots that are near to a given polynomial (Mosier)

Aim of the talk

For a given polynomial with real coefficients, it makes sense to compute both complex and real pseudozero sets even if the latter may be closer to the physical problem the polynomial represent. This is the case when the polynomial coefficients describe

- non-complex physical values as for example in transfer function for control theory
- real perturbations comes from finite precision computation since the rounding error in real coefficients represented by fixed or floating numbers is always a non-complex perturbation

$\Longrightarrow$ The aim of this paper is to compare these two pseudozero sets and evaluate which one is the more convenient and the easiest to compute.

Outline of the talk

1 — Complex pseudozero set

- Definition
- Computation

2 — Real pseudozero set

- Definition
- Computation

3 — Numerical simulations and comparisons

Complex pseudozeros : definition, computation

Complex pseudozero set : definition

Let $p = \sum_{i=0}^{n} p_i z^i$ be a given polynomial of $\mathbf{C}_n[z]$
$\|p\| = \left(\sum_{i=0}^{n} |p_i|^2\right)^{1/2}$

Perturbation :
Neighborhood of polynomial p

$$N_\varepsilon(p) = \{\widehat{p} \in \mathbf{C}_n[z] : \|p - \widehat{p}\| \leqslant \varepsilon\}.$$

Definition of the ε-pseudozero set :

$$Z_\varepsilon(p) = \{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\}.$$

Complex pseudozero set : the set of the zeros of complex polynomials "near p".

Complex pseudozeros are easily computable

Theorem [Trefethen and Toh] :
The complex ε-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|\underline{z}\|} \leqslant \varepsilon \right\},$$

where $\underline{z} = (1, z, \ldots, z^n)$

Pseudozero set : algorithm of computation

1. We mesh a square containing all the roots of p (MATLAB command : `meshgrid`).
2. We compute $g(z) := \frac{|p(z)|}{\|\underline{z}\|}$ for all the nodes z of the grid.
3. We plot the contour level $|g(z)| = \varepsilon$ (MATLAB command : `contour`).

Initialization :

- Find a square containing all the roots of p and all the pseudozeros.
- Find a grid step that separates all the roots.

Complexity of drawing pseudozero set

Let L be the length of the square and h the step of discretization. The evaluation of $g(z) = \frac{|p(z)|}{\|\underline{z}\|}$ needs

- the evaluation of polynomial p, that can be done in $\mathcal{O}(n)$,
- the computation of the 2-norm of a vector that can be done in $\mathcal{O}(n)$.

The complexity of the algorithm to draw the pseudozero set is

$$\boxed{\mathcal{O}((L/h)^2 n)}.$$

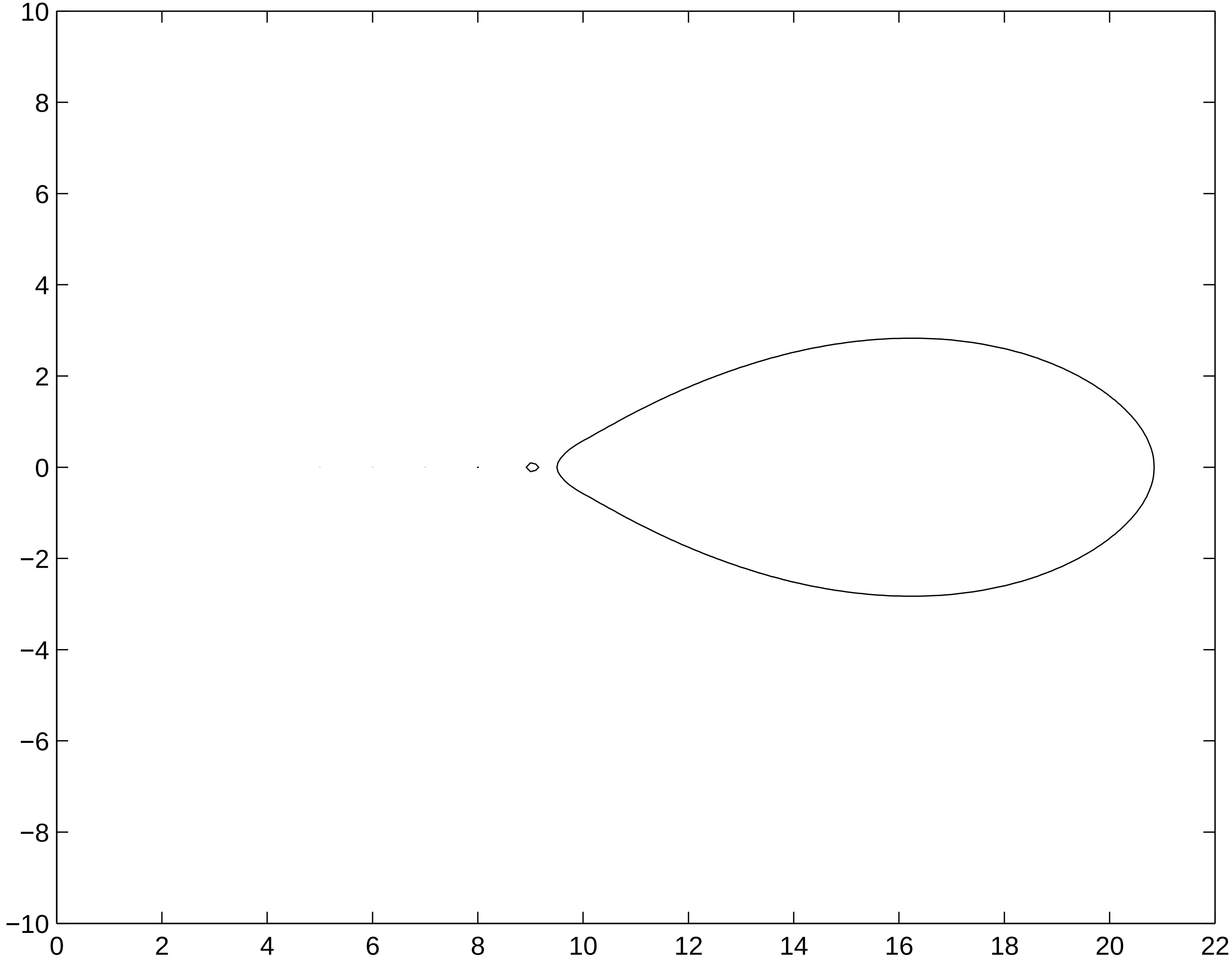
L and h depend on n but also on the polynomial coefficients.

A famous example

Pseudozero set of the *Wilkinson* polynomial

$$\begin{aligned} W_{20} &= (z-1)(z-2)\cdots(z-20), \\ &= z^{20} - 210z^{19} + \cdots + 20!. \end{aligned}$$

We perturb the coefficients with $\varepsilon = 2^{-23}$.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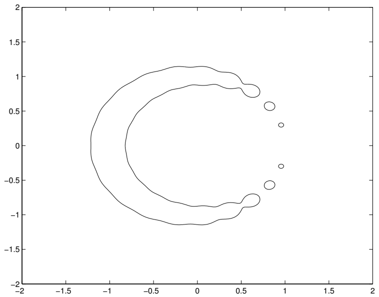
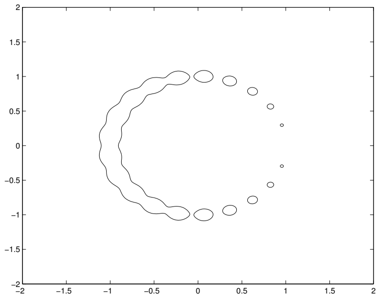
22

Evolution of ε-pseudozero w.r.t ε

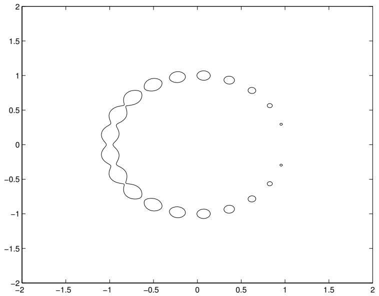
Pseudozero set of the polynomial $p(z) = 1 + z + \cdots + z^{20}$ for different values of ε.



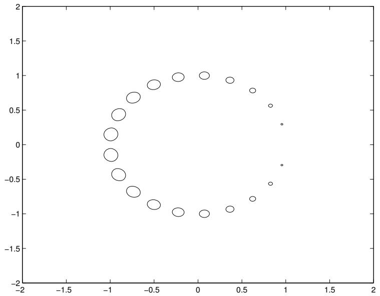
(a) $\varepsilon = 10^{-1}$



(b) $\varepsilon = 10^{-1.2}$



(c) $\varepsilon = 10^{-1.3}$



(d) $\varepsilon = 10^{-1.4}$

Pseudozeros : brief survey of existing references

- Mosier (1986) : Definition and study for the ∞-norm.
- Hinrichsen and Kelb (1993) : Spectral value sets.
- Trefethen and Toh (1994) : Study for the 2-norm.

  pseudozeros $\approx$ pseudospectra of the companion matrix.
- Zhang (2001) : Study the influence of the basis for the 2-norm (condition number of the evaluation).
- Stetter (2004) : *Numerical Polynomial Algebra* (SIAM). Generalization of the previous works.

Real pseudozeros : definition, computation

Real pseudozero set : definition

Let $p = \sum_{i=0}^{n} p_i z^i$ be a given polynomial of $\mathbf{R}_n[z]$
$\|p\| = \left(\sum_{i=0}^{n} |p_i|^2\right)^{1/2}$

Perturbation :
Neighborhood of polynomial p

$$N_\varepsilon^R(p) = \{\widehat{p} \in \mathbf{R}_n[z] : \|p - \widehat{p}\| \leqslant \varepsilon\}.$$

Definition of the ε-pseudozero set :

$$Z_\varepsilon^R(p) = \left\{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon^R(p)\right\}.$$

Real pseudozero set : the set of the zeros of real polynomials "near p".

Real pseudozeros are easily computable

Theorem :
The real ε-pseudozeros set satisfies

$$Z_\varepsilon^R(p) = Z(p) \cup \left\{ z \in \mathbf{C} \backslash Z(p) : h(z) := d(G_R(z), \mathbf{R} G_I(z)) \geqslant \frac{1}{\varepsilon} \right\},$$

where d is defined for $x, y \in \mathbf{R}^{n+1}$,

$$d(x, \mathbf{R}y) = \inf_{\alpha \in \mathbf{R}} \|x - \alpha y\|$$

and $G_R(z)$, $G_I(z)$ are the real and imaginary parts of

$$G(z) = \frac{1}{p(z)} (1, z, \dots, z^n)^T, \ z \in \mathbf{C} \backslash Z(p)$$

Some properties

The function d defined for $x, y \in \mathbf{R}^{n+1}$ by

$$d(x, \mathbf{R}y) = \inf_{\alpha \in \mathbf{R}} \|x - \alpha y\|$$

satisfies

$$d(x, \mathbf{R}y) = \begin{cases} \sqrt{\|x\|^2 - \frac{\langle x, y\rangle}{\|y\|^2}} & \text{if } y \neq 0, \\ \|x\| & \text{if } y = 0 \end{cases}$$

Proposition :
The real ε-pseudozero set $Z_\varepsilon^R(p)$ is symmetric with respect to the real axis.

Pseudozero set : algorithm of computation

1. We mesh a square containing all the roots of p ($\text{M}\textsc{atlab}$ command : `meshgrid`).
2. We compute $h := d(G_R(z), \mathbf{R}G_I(z))$ for all the nodes z of the grid.
3. We plot the contour level $|h(z)| = \frac{1}{\varepsilon}$ ($\text{M}\textsc{atlab}$ command : `contour`).

Initialization :

- Find a square containing all the roots of p and all the pseudozeros.
- Find a grid step that separates all the roots.

Numerical simulations and comparisons

Useful results

Proposition :
Let p be a polynomial of $\mathbf{R}_n[z]$. Then the real pseudozero set is included in the complex pseudozero set, *i.e.*,

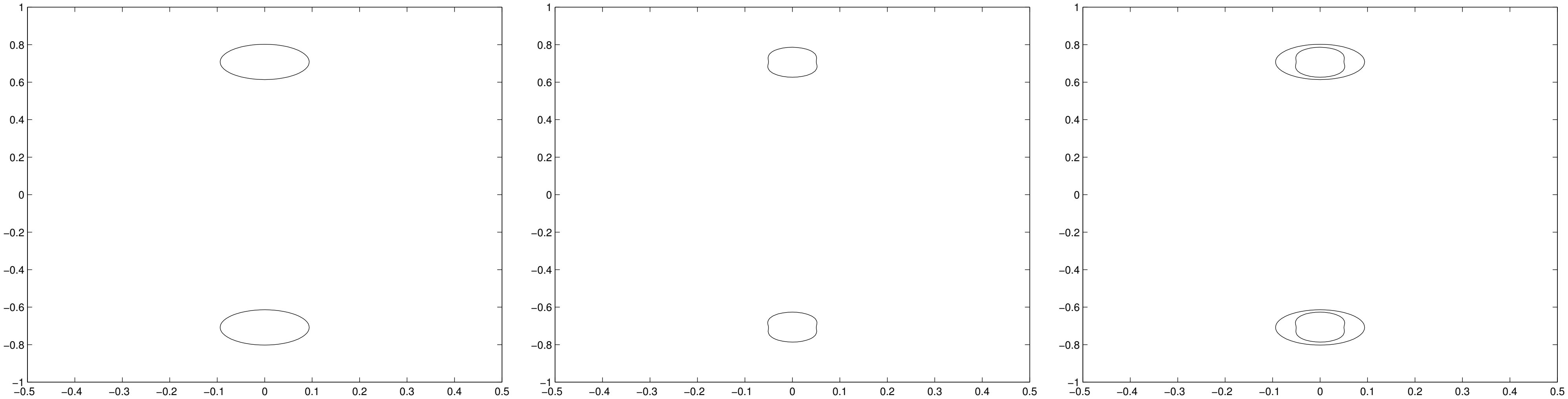
$$Z_\varepsilon^R(p) \subset Z_\varepsilon(p).$$

Lemma [Hinrichsen and Kelb] :
The function

$$d : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}_+, \quad (x, y) \mapsto d(x, \mathbf{R}y)$$

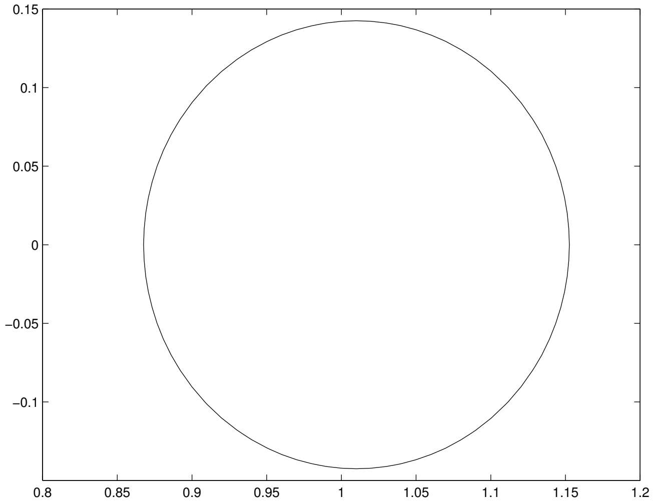
is continuous at all pairs (x, y) with $y \neq 0$ or $x = 0$ and discontinuous at all pairs $(x, 0) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, $x \neq 0$.

Comparison of the complex and real pseudozero sets for $p(z) = 1/2 + z^2$

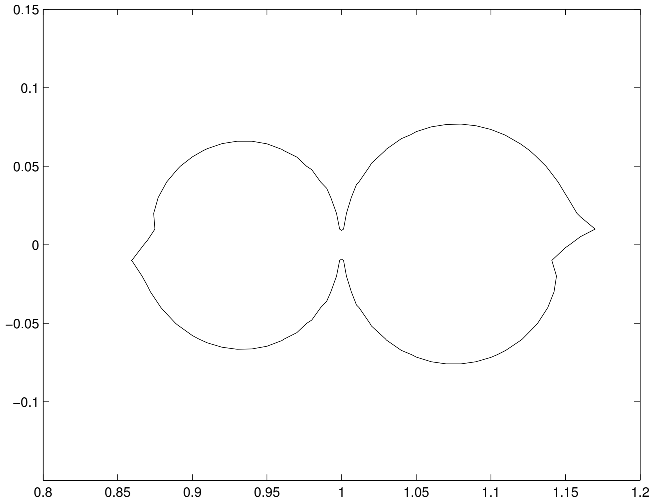


(e) $Z_\varepsilon(p)$ with $\varepsilon = 0.1$ (f) $Z_\varepsilon^R(p)$ with $\varepsilon = 0.1$ (g) Both on the same figure

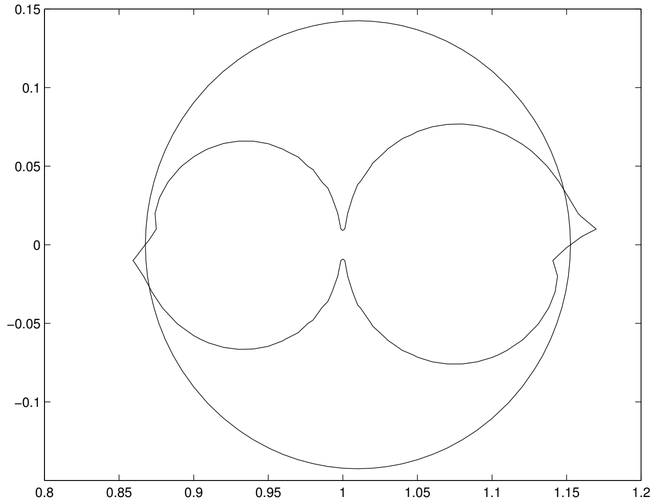
Comparison of the complex and real pseudozero sets for $q(z) = z - 1$



(h) $Z_\varepsilon(q)$ with $\varepsilon = 0.1$



(i) $Z_\varepsilon^R(q)$ with $\varepsilon = 0.1$



(j) Both on the same figure

Conclusion and future work

Conclusion :

We have presented

- a computable formula for the real pseudozero set.
- that computing real pseudozero set may yield inappropriate results

Future work :

- same comparison with matrix polynomials