Pseudozero set decides on polynomial stability

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Motivations

Polynomial coefficients are often approximate values.

Three well known sources of approximation are considered in scientific computation:

1. errors due to discretization and truncation,
2. errors due to roundoff, and
3. errors due to uncertainty in the data.

Use tools designed for such approximate polynomials in control theory.
1 — Pseudozero set

- Definition
- Computation

2 — Applications of pseudozeros in control theory

- Robust stability for polynomials
- Stability radius for polynomials
Pseudozero set : definition

Let $p$ be a given polynomial of $\mathbb{C}_n[z]$

Perturbation :
Neighborhood of polynomial $p$

$$N_\varepsilon(p) = \{ \hat{p} \in \mathbb{C}_n[z] : \|p - \hat{p}\| \leq \varepsilon \}.$$ 

Definition of the $\varepsilon$-pseudozero set :

$$Z_\varepsilon(p) = \{ z \in \mathbb{C} : \hat{p}(z) = 0 \text{ for } \hat{p} \in N_\varepsilon(p) \}.$$ 

$\| \cdot \|$ a norm on the vector of the coefficients of $p$

Pseudozero set : the set of the zeros of polynomials “near $p$”. 
Pseudozeros are easily computable

**Theorem [Stetter]:**
The $\varepsilon$-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|z\|_*} \leq \varepsilon \right\},$$

where $z = (1, z, \ldots, z^n)$ and $\| \cdot \|_*$ is the dual norm of $\| \cdot \|$, 

$$\|y\|_* = \sup_{x \neq 0} \frac{|y^* x|}{\|x\|}$$
Pseudozero set : algorithm of computation

1. We mesh a square containing all the roots of $p$ (**MATLAB** command : \texttt{meshgrid}).

2. We compute $g(z) := \frac{|p(z)|}{\|z\|_*}$ for all the nodes $z$ of the grid.

3. We plot the contour level $|g(z)| = \varepsilon$ (**MATLAB** command : \texttt{contour}).

**Initialization :**

- Find a square containing all the roots of $p$ and all the pseudozeros.
- Find a grid step that separates all the roots.
A famous example

Pseudozero set of the *Wilkinson* polynomial

\[ W_{20} = (z - 1)(z - 2) \cdots (z - 20), \]
\[ = z^{20} - 210z^{19} + \cdots + 20!. \]

We only perturb the coefficient of \( z^{19} \) with \( \varepsilon = 2^{-23} \).
One uses the weighted-norm \( \| \cdot \|_\infty \):

\[ \|p\|_\infty = \max_i \left| \frac{p_i}{m_i} \right| \text{ with } m_i \text{ non negative} \]

with \( m_{19} = 1, \) \( m_i = 0 \) otherwise and the convention \( m/0 = \infty \) if \( m > 0 \) and \( 0/0 = 0 \).
Evolution of $\varepsilon$-pseudozero w.r.t $\varepsilon$

Pseudozero set of the polynomial $p(z) = 1 + z + \cdots + z^{20}$ for different values of $\varepsilon$.

(a) $\varepsilon = 10^{-1}$

(b) $\varepsilon = 10^{-1.2}$

(c) $\varepsilon = 10^{-1.3}$

(d) $\varepsilon = 10^{-1.4}$
Pseudozeros : brief survey of existing references

▶ Mosier (1986) : Definition and study for the $\infty$-norm.
▶ Trefethen and Toh (1994) : Study for the 2-norm.
  pseudozeros $\approx$ pseudospectra of the companion matrix.
▶ Zhang (2001) : Study the influence of the basis for the 2-norm (condition number of the evaluation).
Other applications of pseudozero set: Robust stability and Stability radius for polynomials
Schur robust stability in control theory

Schur stability: $|\text{roots of } p| < 1.$

$\varepsilon$-pseudozero set of $p(z) = (z - 0.8)^2$ for $\varepsilon = 0.1$ and $\varepsilon = 0.01$. 

![Diagram showing the unit circle and the $\varepsilon$-pseudozero sets for different values of $\varepsilon$.]
Hurwitz robust stability in control theory

Hurwitz stability : Real part of roots of $p < 0$.

$\varepsilon$-pseudozero set of $p(z) = (z + 1)^2$ for $\varepsilon = 0.4$. 
Computation of stability radius

\( \mathcal{P}_n \): polynomials of \( \mathbb{C}[X] \) of degree at most \( n \)

\( \mathcal{M}_n \): monic polynomials of \( \mathcal{P}_n \) of degree \( n \)

\( \| \cdot \| \): the 2-norm of the coefficients of a polynomial

**Definition.** A polynomial is stable if all its roots have negative real part and unstable otherwise (Hurwitz stability).

The function *abscissa* \( a : \mathcal{P} \to \mathbb{R} \) is defined by

\[
a(p) = \max\{\text{Re}(z) : p(z) = 0\}.
\]

A polynomial \( p \) is stable \( \iff \) \( a(p) < 0 \)
In control theory, transfer functions are often written as $H(p) = \frac{N(p)}{D(p)}$ where $N$ and $D$ are polynomials.

The system is stable if $D$ is a stable polynomial.

Question: if $D$ is stable, how far is it from an unstable system?

Problem: Find the distance to the nearest unstable system.

(we assume that $D$ is monic)
How to compute the stability radius

**Stability radius** $\beta(p)$: distance of the polynomial $p \in \mathcal{M}_n$ from the set of monic unstable polynomials.

$$\beta(p) = \min\{\|p - q\| : q \in \mathcal{M}_n \text{ and } a(q) \geq 0\}.$$  

**Statement of the problem:**

Given a polynomial $p \in \mathcal{M}_n$, let us compute $\beta(p)$. 

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Our solution

Tools
• an explicit formula that defines the pseudozeros
• the continuous dependency of the roots w.r.t the polynomial coefficients
• Sturm sequences to count the real roots

The results
• an algorithm that approximates $\beta(p)$ up to an arbitrary accuracy $\tau$
• a plot showing the pseudozeros at the distance $\beta(p)$
  → a qualitative analysis of the result
  → a visualization of the result
Pseudozero set for monic polynomials

**Perturbation** : Neighborhood of polynomial \( p \)

\[
N_\varepsilon(p) = \{ \hat{p} \in \mathcal{M}_n : \| p - \hat{p} \| \leq \varepsilon \}.
\]

**Definition of the \( \varepsilon \)-pseudozero set** :

\[
Z_\varepsilon(p) = \{ z \in \mathbb{C} : \hat{p}(z) = 0 \text{ for } \hat{p} \in N_\varepsilon(p) \}.
\]

\( \| \cdot \| \) is the 2-norm on the vector of the coefficients of \( p \)

The \( \varepsilon \)-pseudozeros set satisfies

\[
Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\| z \|} \leq \varepsilon \right\},
\]

where \( z = (1, z, \ldots, z^{n-1}) \)
Another characterization of $Z_\varepsilon(p)$

Let us denote $h_{p,\varepsilon} : \mathbb{R}^2 \rightarrow \mathbb{R}$, the function

$$h_{p,\varepsilon}(x, y) = |p(x + iy)|^2 - \varepsilon^2 \sum_{j=0}^{n-1} (x^2 + y^2)^j.$$

Then one has

$$Z_\varepsilon(p) = \{(x, y) \in \mathbb{R}^2 : h_{p,\varepsilon}(x, y) \leq 0\}$$

$\implies h_\varepsilon(\cdot, y)$ et $h_\varepsilon(x, \cdot)$ are polynomials of degree $2n$.

**Theorem.** The equation $h_{p,\varepsilon}(0, y) = 0$ has a real solution $y$ if and only if $\beta(p) \leq \varepsilon$.  

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Algorithm (bisection)

Require : a stable polynomial $p$ and a tolerance $\tau$
Ensure : a number $\alpha$ such that $|\alpha - \beta(p)| \leq \tau$

1: $\gamma := 0$, $\delta := \|p - z^n\|$  
2: while $|\gamma - \delta| > \tau$ do  
3: \hspace{1em} $\varepsilon := \frac{\gamma + \delta}{2}$  
4: \hspace{1em} if the equation $h_{p,\varepsilon}(0, y) = 0$ has a real solution then  
5: \hspace{2em} $\delta := \varepsilon$  
6: \hspace{1em} else  
7: \hspace{2em} $\gamma := \varepsilon$  
8: \hspace{1em} end if  
9: \hspace{1em} end while  
10: return $\alpha = \frac{\gamma + \delta}{2}$
Numerical simulation

For \( p(z) = z + 1 \), the algorithm gives \( \beta(p) \approx 0.999996 \)

**Fig. 1:** \( \beta(p) \)-pseudozero set of \( p(z) = z + 1 \)
Numerical simulation (contd)

For $p(z) = z^2 + z + 1/2$, the algorithm gives $\beta(p) \approx 0.485868$

**Fig. 2**: $\beta(p)$-pseudozero set of $p(z) = z^2 + z + 1/2$
For \( p(z) = z^3 + 4z^2 + 6z + 4 \), the algorithm gives \( \beta(p) \approx 2.610226 \).
Conclusion

Pseudozero set provides
- a better understanding of the effect of coefficient perturbations
- some applications for robust stability