Some Results on Structured Pseudospectra

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Outline

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Why structured matrices?

- Structured matrices are used in various fields such as signal processing, etc.
- Using the structure of a matrix, we get some better properties
- Substantial interest in algorithms for structured problems in recent years
- Growing interest in structured perturbation analysis
- In general perturbation and error analysis for structured solvers are performed with *general* perturbations: for a structured solver nothing else but structured perturbations are *possible*

Our structures

Toeplitz matrices $(t_{i-j})_{i,j=0}^{n-1}$	$\begin{pmatrix} t_0 & t_{-1} & \cdots & t_{1-n} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & t_1 & t_0 \end{pmatrix}$
Hankel matrices $(h_{i,j})_{i,j=0}^{n-1}$	$\begin{pmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \iddots & h_n \\ \vdots & \iddots & \iddots & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n-2} \end{pmatrix}$
Circulant matrices $(v_i)_{i=0}^{n-1}$	$\begin{pmatrix} v_0 & v_{n-1} & \cdots & v_1 \\ v_1 & v_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_{n-1} \\ v_{n-1} & \cdots & v_1 & v_0 \end{pmatrix}$

Number of independant parameters

- In the following table, k represents the number of independant parameters for the different structures

Structure	general	Toeplitz	circulant	Hankel
k	n^2	$2n-1$	n	$2n-1$

Notations

In this talk, we will use the following notation:

struct	Toeplitz, circulant or Hankel
$M_n(\mathbf{C})$	set of complex $n \times n$ matrices
$M_n^{\mathrm{struct}}(\mathbf{C})$	set of structured complex $n \times n$ matrices
$\lVert \cdot \rVert$	spectral norm
I, I_n	identity matrix (with n rows and columns)
$\sigma_{\min}(A)$	smallest singular value of A
$\Lambda(A)$	spectrum of A

Definition of pseudospectra

The ε-pseudospectrum of a matrix A, denoted $\Lambda_\varepsilon(A)$, is the subset of complex numbers consisting of all eigenvalues of all complex matrices within a distance ε of A

Definition

For a real $\varepsilon > 0$, the ε-pseudospectrum of a matrix $A \in M_n(\mathbf{C})$ is the set

$$\Lambda_\varepsilon(A) = \{z \in \mathbf{C} : z \in \Lambda(X) \text{ where } X \in M_n(\mathbf{C}) \text{ and } \|X - A\| \leq \varepsilon\}.$$

Distance to singularity

Definition

Given a nonsingular matrix $A \in M_n(\mathbf{C})$, we define the distance to singularity by

$$d(A) = \min\{\|\Delta A\| : A + \Delta A \text{ singular}, \Delta A \in M_n(\mathbf{C})\}.$$

Lemma

Let nonsingular $A \in M_n(\mathbf{C})$. Then we have

$$d(A) = \|A^{-1}\|^{-1}.$$

Characterisation of pseudospectra

Theorem

The following assertions are equivalent

(i) $\Lambda_\varepsilon(A)$ is the ε-pseudospectrum of a matrix A

(ii) $\Lambda_\varepsilon(A) = \{z \in \mathbf{C} : \|(zI - A)^{-1}\| \geq \varepsilon^{-1}\}$

(iii) $\Lambda_\varepsilon(A) = \{z \in \mathbf{C} : \sigma_{\min}(zI - A)\| \leq \varepsilon\}$

(iv) $\Lambda_\varepsilon(A) = \{z \in \mathbf{C} : d(zI - A) \leq \varepsilon\}$

Definition of structured pseudospectra

The structured ε-pseudospectrum of a matrix A, denoted $\Lambda_\varepsilon^{\text{struct}}(A)$, is the subset of complex numbers consisting of all eigenvalues of all complex structured matrices within a distance ε of A

Definition

For a real $\varepsilon > 0$, the structured ε-pseudospectrum of a matrix $A \in M_n^{\text{struct}}(\mathbf{C})$ is the set

$$\Lambda_\varepsilon^{\text{struct}}(A) = \{z \in \mathbf{C} : z \in \Lambda(X) \text{ where } X \in M_n^{\text{struct}}(\mathbf{C}) \text{ and } \|X - A\| \leq \varepsilon\}.$$

Structured distance to singularity

Definition

Given a nonsingular matrix $A \in M_n^{\mathrm{struct}}(\mathbf{C})$, we define the structured distance to singularity by

$$d^{\mathrm{struct}}(A) = \min\{\|\Delta A\| : A + \Delta A \text{ singular}, \Delta A \in M_n^{\mathrm{struct}}(\mathbf{C})\}.$$

Theorem (Rump [8, Thm 12.2])

Let nonsingular $A \in M_n^{\mathrm{struct}}(\mathbf{C})$ with *struct* being Toeplitz , Hankel or circulant. Then we have

$$d^{\mathrm{struct}}(A) = d(A) = \|A^{-1}\|^{-1}.$$

Characterisation of structured pseudospectra

Lemma

Given $\varepsilon > 0$ and $A \in M_n^{\mathrm{struct}}(\mathbf{C})$ with *struct* Toeplitz or circulant, the structured ε-pseudospectrum satisfies

$$\Lambda_\varepsilon^{\mathrm{struct}}(A) = \{z \in \mathbf{C} : d^{\mathrm{struct}}(A - zI) \leq \varepsilon\}.$$

Theorem

Given $\varepsilon > 0$ and $A \in M_n^{\mathrm{struct}}(\mathbf{C})$ with *struct* Toeplitz or circulant, the ε-pseudospectrum and the structured ε-pseudospectrum satisfy

$$\Lambda_\varepsilon^{\mathrm{struct}}(A) = \Lambda_\varepsilon(A).$$

What for others linear structures?

We do not have equality for Hermitian and skew-Hermitian structures.
For example for Hermitian structure we always have $\Lambda_\varepsilon^{\mathrm{herm}}(A) \subsetneq \mathbf{R}$ whereas one can find an Hermitian matrix such that $\Lambda_\varepsilon(A) \not\subseteq \mathbf{R}$.

The polynomial eigenvalue problem

Problem

Find the solutions $(x, \lambda) \in \mathbf{C}^n \times \mathbf{C}$ of

$$P(\lambda)x = 0,$$

where

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n(\mathbf{C})$, $k = 0 : m$

If $x \neq 0$ then λ is called an eigenvalue and x the corresponding eigenvector. The set of eigenvalues of P is denoted $\Lambda(P)$. We assume that P has only finite eigenvalues (and pseudoeigenvalues)

Definition of pseudospectra

Let us define

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n(\mathbf{C})$.

Definition

For a given $\varepsilon > 0$, the ε-pseudospectrum of P is the set

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbf{C} : (P(\lambda) + \Delta P(\lambda))x = 0 \text{ for some } x \neq 0 \\ \text{with } \|\Delta A_k\| \leq \alpha_k \varepsilon, k = 0 : m\}.$$

The nonnegative parameters $\alpha_1, \ldots, \alpha_m$ allow freedom in how perturbations are measured

Characterisation of pseudospectra

Lemma (Tisseur and Higham [9])

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbf{C} : d(P(\lambda)) \le \varepsilon p(|\lambda|)\},$$

where $p(x) = \sum_{k=0}^{m} \alpha_k x^k$.

Definition of structured pseudospectra

We suppose that ΔA_k have a structure belonging to struct. We also suppose that all the matrices A_k and ΔA_k, $k = 0 : n$, belong to $M_n^{\mathrm{struct}}(\mathbf{C})$ for a given structure struct. Let

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n^{\mathrm{struct}}(\mathbf{C})$, $k = 0 : m$ and

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n^{\mathrm{struct}}(\mathbf{C})$. $P(\lambda)$ and $\Delta P(\lambda)$ belong to $M_n^{\mathrm{struct}}(\mathbf{C})$.

Definition

We define the structured ε-pseudospectrum of P by

$$\Lambda_\varepsilon^{\mathrm{struct}}(P) = \{\lambda \in \mathbf{C} : (P(\lambda) + \Delta P(\lambda))x = 0 \text{ for some } x \neq 0 \\ \text{with } \Delta A_k \in M_n^{\mathrm{struct}}(\mathbf{C}), \|\Delta A_k\| \leq \alpha_k \varepsilon, k = 0 : n\}.$$

Characterisation of structured pseudospectra

Lemma

For $\mathrm{struct} \in \{\mathrm{Toep}, \mathrm{circ}, \mathrm{Hankel}\}$, we have

$$\Lambda_\varepsilon^{\mathrm{struct}}(P) = \{\lambda \in \mathbf{C} : d^{\mathrm{struct}}(P(\lambda)) \leq \varepsilon p(|\lambda|)\},$$

where $p(x) = \sum_{k=0}^{n} \alpha_k x^k$.

Theorem

Given $\varepsilon > 0$ and $P(\lambda) \in M_n^{\mathrm{struct}}(\mathbf{C})$ a matrix polynomial with $\mathrm{struct} \in \{\mathrm{Toep}, \mathrm{circ}, \mathrm{Hankel}\}$, the ε-pseudospectrum and the structured ε-pseudospectrum satisfy

$$\Lambda_\varepsilon^{\mathrm{struct}}(P) = \Lambda_\varepsilon(P).$$

Real structured perturbations

Consider

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n(\mathbf{R})$, $k = 0 : m$ and

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n(\mathbf{R})$. Suppose that $P(\lambda)$ is subject to structured perturbations:

$$[\Delta A_0, \ldots, \Delta A_m] = D\Theta[E_0, \ldots, E_m],$$

with $D \in M_{n,1}(\mathbf{R})$, $\Theta \in M_{1,t}(\mathbf{R})$ and $E_k \in M_{t,n}(\mathbf{R})$, $k = 0 : m$.
For notational convenience, we introduce

$$E(\lambda) = E[I_n, \lambda I_n, \ldots, \lambda^m I_n]^T = \lambda^m E_m + \lambda^{m-1} E_{m-1} + \cdots + E_0,$$

and

$$G(\lambda) = E(\lambda)P(\lambda)^{-1}D = G_R(\lambda) + iG_I(\lambda), \quad G_R(\lambda), G_I(\lambda) \in \mathbf{R}^t.$$

Definition and characterisation of pseudospectra

Definition

The structured ε-pseudospectrum is defined by

$\Lambda_\varepsilon(P) = \{\lambda \in \mathbf{C} : (P(\lambda) + D\Theta E(\lambda))x = 0 \text{ for some } x \neq 0, \|\Theta\| \leq \varepsilon\}$

We denote for $x, y \in \mathbf{R}^t$,

$$d(x, \mathbf{R}y) = \inf_{\alpha \in \mathbf{R}} \|x - \alpha y\|,$$

the distance of the point x from the linear subspace $\mathbf{R}y = \{\alpha y, \alpha \in \mathbf{R}\}$.

Theorem

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbf{C} \setminus \Lambda(P) : d(G_R(\lambda), \mathbf{R}G_I(\lambda)) \geq 1/\varepsilon\} \cup \Lambda(P)$$

Conclusion

We have

- The structured pseudospectrum is equal to the pseudospectrum for the two following structures: Toeplitz and circulant
- This result is false for structures Hermitian and skew-Hermitian
- We have generalized these results to pseudospectra of matrix polynomials.
- We have given a formula for structured pseudospectra of real matrix polynomials

References I

A. Böttcher, M. Embree, and V. I. Sokolov.
On large Toeplitz band matrices with an uncertain block.
Linear Algebra Appl., 366:87–97, 2003.
Special issue on structured matrices: analysis, algorithms and applications (Cortona, 2000).

A. Böttcher and S. Grudsky.
Spectral Properties of Banded Toeplitz Matrices.
Book to appear.

References II

A. Böttcher, S. Grudsky, and A. Kozak.
On the distance of a large Toeplitz band matrix to the nearest singular matrix.
In *Toeplitz matrices and singular integral equations (Pobershau, 2001)*, volume 135 of *Oper. Theory Adv. Appl.*, pages 101–106. Birkhäuser, Basel, 2002.

Nicholas J. Higham and Françoise Tisseur.
More on pseudospectra for polynomial eigenvalue problems and applications in control theory.
Linear Algebra Appl., 351/352:435–453, 2002.

D. Hinrichsen and B. Kelb.
Spectral value sets: a graphical tool for robustness analysis.
Systems Control Lett., 21(2):127–136, 1993.

References III

D. Hinrichsen and A. J. Pritchard.
Robustness measures for linear systems with application to stability radii of Hurwitz and Schur polynomials.
Internat. J. Control, 55(4):809–844, 1992.

Peter Lancaster and Panayiotis Psarrakos.
On the pseudospectra of matrix polynomials.
Numerical Analysis Report No. 445, Manchester Centre for Computational Mathematics, Manchester, England, February 2004.

Siegfried M. Rump.
Structured perturbations. I. Normwise distances.
SIAM J. Matrix Anal. Appl., 25(1):1–30 (electronic), 2003.

References IV

Françoise Tisseur and Nicholas J. Higham.
Structured pseudospectra for polynomial eigenvalue problems, with applications.
SIAM J. Matrix Anal. Appl., 23(1):187–208 (electronic), 2001.

L. N. Trefethen.
Pseudospectra of matrices.
In *Numerical analysis 1991 (Dundee, 1991)*, volume 260 of *Pitman Res. Notes Math. Ser.*, pages 234–266. Longman Sci. Tech., Harlow, 1992.