Validated Pseudozero Set of Polynomials

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Abstract

The pseudozero set (**[2-4]**) of a polynomial p is the set of complex numbers that are roots of polynomials which are near to p. This is a powerful tool to analyze the sensitivity of roots with respect to perturbations of the coefficients. Some applications in algebraic computation and robust control theory have been proposed recently.

The set $\mathcal{P}_n$ denotes the set of polynomials with complex coefficients and degree at most n. Let $p \in \mathcal{P}_n$ given by

$$p(z) = p_0 + p_1 z + \cdots + p_n z^n.$$

Representing polynomial p by the vector of its coefficients, we choose the norm $\|\cdot\|$ on $\mathcal{P}_n$ being some norm on $\mathbf{C}^{n+1}$ of the polynomial coefficient vector. For this norm, we define an ε-neighborhood of p to be the set of every polynomial of degree at most n, closed enough to p, that is,

$$N_\varepsilon(p) = \{\widehat{p} \in \mathcal{P}_n : \|p - \widehat{p}\| \leq \varepsilon\}.$$

Then the ε-pseudozero set of p is defined to include all the zeros of the ε-neighborhood of p. A non constructive definition of this set is

$$Z_\varepsilon(p) = \{z \in \mathbf{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\}.$$

An explicit formula to compute this set is given (**[3]**) by

$$Z_\varepsilon(p) = \left\{ z \in \mathbf{C} : \frac{|p(z)|}{\|\underline{z}\|_*} \leq \varepsilon \right\},$$

where $\underline{z} = (1, z, \dots, z^n)$ and $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$,

$$\|y\|_* = \sup_{x \neq 0} \frac{|y^* x|}{\|x\|}.$$

The drawing of this set is often done using a level contour command like `contour` in MATLAB. In this case, we cannot certify the drawing. Our aim is to present a robust algorithm to draw certify drawing of the set. For this, we use algorithms from **[1]** to draw inner and outer approximations of

that set using interval arithmetic. We also present a graphical MATLAB interface to draw such sets.

An *interval polynomial* is a polynomial whose coefficients are real intervals. An interval polynomial of degree n can be written as

$$p(z) = \sum_{i=0}^{n} [a_i, b_i] z^i.$$

The zeros of the interval polynomial is the set (denoted $\mathbf{Z}(p)$) defined by

$$\mathbf{Z}(p) := \{z \in \mathbf{C} : \text{there exist } m_i \in [a_i, b_i], i = 0 : n \text{ such that} \sum_{i=0}^{n} m_i z^i = 0\}.$$

Thanks to the theory of pseudozero set, it is possible to derive an explicit formula to compute the set $\mathbf{Z}(p)$ and so to derive an algorithm to compute certified drawing of the zeros of interval polynomials.

References:

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