Approximate polynomial problems
and associated tools

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11th GAMM- IMACS International Symposium
SCAN 2004

Kyushu University, October 4 - 8, 2004, Fukuaka, Japan
Motivations

Polynomial coefficients are often approximate values.

Three well known sources of approximation are considered in scientific computation:

(1) errors due to uncertainty in the data,
(2) errors due to discretization and truncation, and
(3) errors due to finite precision roundoff.

We need tools designed for such approximate polynomials in computer algebra, control theory, etc.
Outline of the talk

1 — Pseudozero set
• Definition and computation

2 — Pseudozeros and polynomial primality
• Some definitions
• Contribution of pseudozero set

3 — Other applications of pseudozeros
• Robust stability in control theory
• Stability radius for polynomials
• Multiplicity of polynomial roots
Pseudozeros: definition, computation and motivation
Pseudozero set : definition

Let $p$ be a given polynomial of $\mathbb{C}_n[z]$

**Perturbation :**
Neighborhood of polynomial $p$

$$N_\varepsilon(p) = \{\hat{p} \in \mathbb{C}_n[z] : \|p - \hat{p}\| \leq \varepsilon\}.$$ 

**Definition of the $\varepsilon$-pseudozero set :**

$$Z_\varepsilon(p) = \{z \in \mathbb{C} : \hat{p}(z) = 0 \text{ for } \hat{p} \in N_\varepsilon(p)\}.$$ 

$\| \cdot \|$ a norm on the vector of the coefficients of $p$

Pseudozero set : the set of the zeros of polynomials “near $p$”.
Pseudozeros : brief survey of main references

- Mosier (1986) : Definition and study form the $\infty$-norm.
- Trefethen and Toh (1994) : Study for the $2$-norm. 
  \[\text{pseudozeros} \approx \text{pseudospectra of the companion matrix}.\]
- Zhang (2001) : Study of the influence of the basis for the $2$-norm (condition number of the evaluation).
Pseudozeros are easily computable

**Theorem**: (Mosier, Toh and Trefethen, Stetter)
The $\varepsilon$-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|z\|_*} \leq \varepsilon \right\},$$

where $z = (1, z, \ldots, z^n)$ and $\| \cdot \|_*$ is the dual norm of $\| \cdot \|$, 

$$\|y\|_* = \sup_{x \neq 0} \frac{|y^* x|}{\|x\|}$$
Algorithm of computation

Algorithm to draw the $\varepsilon$-pseudozero set:

1. We mesh a square containing all the roots of $p$ (MATLAB command: meshgrid).
2. We compute $g(z) := \frac{|p(z)|}{\|z\|_*}$ for all the nodes $z$ in the grid.
3. We draw the contour level $|g(z)| = \varepsilon$ (MATLAB command: contour).

Problems:

- Find a square containing all the roots of $p$ and all the pseudozeros.
- Find a grid step that separates all the roots.
A famous example

Pseudozero set of the *Wilkinson* polynomial

\[ W_{20} = (z - 1)(z - 2) \cdots (z - 20), \]
\[ = z^{20} - 210z^{19} + \cdots + 20!. \]

We perturb only the coefficient of \( z^{19} \) with \( \epsilon = 2^{-23} \).

One uses the weighted-norm \( \| \cdot \|_{\infty} \):

\[ \| p \|_{\infty} = \max_i \frac{|p_i|}{m_i} \text{ with } m_i \text{ non negative} \]

with \( m_{19} = 1, m_i = 0 \) otherwise and the convention \( m/0 = \infty \) if \( m > 0 \) and \( 0/0 = 0 \).
Evolution of $\varepsilon$-pseudzero w.r.t. $\varepsilon$

Pseud zero set of the polynomial $p(z) = 1 + z + \cdots + z^{20}$ for different values of $\varepsilon$.

(a) $\varepsilon = 10^{-1}$

(b) $\varepsilon = 10^{-1.2}$

(c) $\varepsilon = 10^{-1.3}$

(d) $\varepsilon = 10^{-1.4}$
Application of pseudozeros to primality
Definition of $\varepsilon$-GCD of polynomials

Let $p$ and $q$ be two polynomials of degree $n$ and $m$ and let $\varepsilon$ be a nonnegative number. We define

- an $\varepsilon$-divisor: a divisor of perturbed polynomials $\hat{p}$ and $\hat{q}$ satisfying $\deg \hat{p} \leq n$, $\deg \hat{q} \leq m$ and $\max(\|p - \hat{p}\|, \|q - \hat{q}\|) \leq \varepsilon$.
- an $\varepsilon$-GCD: an $\varepsilon$-divisor of maximal degree.
- Two polynomials $p$ and $q$ are $\varepsilon$-coprime if their $\varepsilon$-GCD equals 1.
Definition of $\varepsilon$-primality

Remarks:
• $\varepsilon$ measures the uncertainty about the coefficients (representing finite precision).
• Uniqueness of the degree but not of the $\varepsilon$-GCD.
• Dependency with respect to the basis field.

Computation:
• Sylvester criterion: algorithm COPRIME [Beckermann and Labahn 1998].
• Zeng (2004): algorithm based on Gauss-Newton on a perjorative manifold
• Graphical: pseudozero set.
Pseudozeros to solve the $\varepsilon$-primality problem

From the definition of the $\varepsilon$-pseudozero set, we derive that

- if the intersection of the $\varepsilon$-pseudozero sets of $p$ and $q$ is empty then the two polynomials are $\varepsilon$-coprime,
- if the intersection is not empty then they are not $\varepsilon$-coprime.
Numerical simulation

- **Input**: $p$ and $q$ two polynomials.
- **Output**: a graphic.
- **Drawbacks**: qualitative tool.
- **Example in** $\| \cdot \|_2$:

  \[
  p = (z - 1)(z - 2) = z^2 - 3z + 2
  \]
  \[
  q = (z - 1.08)(z - 1.82) = z^2 - 2.9z + 1.9656
  \]
\[ p = (z - 1)(z - 2) = z^2 - 3z + 2, \quad q = (z - 1.08)(z - 1.82) = z^2 - 2.9z + 1.9656 \]
\[ p = (z - 1)(z - 2) = z^2 - 3z + 2, \quad q = (z - 1.08)(z - 1.82) = z^2 - 2.9z + 1.9656 \]
Other applications of pseudozeros
Schur robust stability in control theory

Schur stability: \(|\text{roots of } p| < 1\).

\(\varepsilon\)-pseudozero set of \(p(z) = (z - 0.8)^2\) for \(\varepsilon = 0.1\) and \(\varepsilon = 0.01\).
Hurwitz robust stability in control theory

Hurwitz stability: Real part of roots of $p < 0$.

$\varepsilon$-pseudozero set of $p(z) = (z + 1)^2$ for $\varepsilon = 0.4$. 

![Graph of $p(z) = (z + 1)^2$ for $\varepsilon = 0.4$.]
Computation of stability radius

\( \mathcal{P}_n \): polynomials of \( \mathbb{C}[X] \) of degree less or equal than \( n \)

\( \mathcal{M}_n \): monic polynomials of \( \mathcal{P}_n \)

\( \| \cdot \| \): the 2-norm of the coefficients of a polynomial

**Definition.** A polynomial is said to be **stable** if all the roots have negative real part and unstable otherwise (Hurwitz stability).

The function **abscissa** \( a : \mathcal{P} \rightarrow \mathbb{R} \) is defined by \( a(p) = \max\{\text{Re}(z) : p(z) = 0\} \).

A polynomial \( p \) is stable \( \iff \) \( a(p) < 0 \)

**Stability radius** \( \beta(p) \): distance of the polynomial \( p \in \mathcal{M}_n \) from the set of monic unstable polynomials.

\[
\beta(p) = \min\{\|p - q\| : q \in \mathcal{M}_n \text{ and } a(q) \geq 0\}.
\]
Another characterization of $Z_\varepsilon(p)$

Let us denote $h_{p,\varepsilon} : \mathbb{R}^2 \to \mathbb{R}$ the function defined by

$$h_{p,\varepsilon}(x, y) = |p(x + iy)|^2 - \varepsilon^2 \sum_{j=0}^{n-1} (x^2 + y^2)^j.$$  

Then one has

$$Z_\varepsilon(p) = \{ (x, y) \in \mathbb{R}^2 : h_{p,\varepsilon}(x, y) \leq 0 \}$$

$\implies h_\varepsilon(\cdot, y)$ et $h_\varepsilon(x, \cdot)$ are polynomials of degree $2n$.

**Theorem.** The equation $h_{p,\varepsilon}(0, y) = 0$ has a real solution $y$ if and only if $\beta(p) \leq \varepsilon$. 

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Algorithm (bisection)

Require: a stable polynomial \( p \) and a tolerance \( \tau \)
Ensure: a number \( \alpha \) such that \( |\alpha - \beta(p)| \leq \tau \)

1. \( \gamma := 0, \quad \delta := \|p - z^n\| \)
2. \( \textbf{while } |\gamma - \delta| > \tau \textbf{ do} \)
3. \( \varepsilon := \frac{\gamma + \delta}{2} \)
4. \( \textbf{if the equation } h_{p,\varepsilon}(0, y) = 0, \ y \in \mathbb{R} \text{ has a solution then} \)
5. \( \delta := \varepsilon \)
6. \( \textbf{else} \)
7. \( \gamma := \varepsilon \)
8. \( \textbf{end if} \)
9. \( \textbf{end while} \)
10. \( \textbf{return } \alpha = \frac{\gamma + \delta}{2} \)
Numerical simulation

For the polynomial $p(z) = z^2 + z + 1/2$, the algorithm gives $\beta(p) \approx 0.485868$ with $\tau = 0.00001$

**Fig. 1:** $\beta(p)$-pseudozero set of $p(z) = z^2 + z + 1/2$
Multiplicity of polynomial roots

Computation of the $\varepsilon$-pseudozeros of polynomials:

$$p_1(z) = z - 1, \quad p_2(z) = (z - 1)^2, \quad p_3(z) = (z - 1)^3,$$

with, respectively, $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^2$, $\varepsilon_3 = \varepsilon^3$ and $\varepsilon = 10^{-1}$.

(a) $Z_\varepsilon$ of $p_1, p_2, p_3$ and $\varepsilon = 10^{-1}$

(b) Pseudozero sets $Z_\varepsilon(p_1), \quad Z_\varepsilon^2(p_2),\quad Z_\varepsilon^3(p_3)$ for $\varepsilon = 10^{-1}$
Conclusion

The pseudozero set provides

1. a better understanding of the effect of coefficients perturbation;
2. a test for $\varepsilon$-primality of two polynomials;
3. an application for robust stability and multiplicity.