Approximate polynomial problems and associated tools

Stef Graillat and Philippe Langlois
University of Perpignan, France

langlois@univ-perp.fr

http://gala.univ-perp.fr/~langlois

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Motivations

Polynomial coefficients are often approximate values

Three well known sources of approximation are considered in scientific computation :

(1) errors due to uncertainty in the data,

(2) errors due to discretization and truncation, and

(3) errors due to finite precision roundoff.

$\Longrightarrow$ We need tools designed for such approximate polynomials in computer algebra, control theory, etc.

Outline of the talk

1 — Pseudozero set

- Definition and computation

2 — Pseudozeros and polynomial primality

- Some definitions
- Contribution of pseudozero set

3 — Other applications of pseudozeros

- Robust stability in control theory
- Stability radius for polynomials
- Multiplicity of polynomial roots

Pseudozeros : definition, computation and motivation

Pseudozero set : definition

Let p be a given polynomial of $\mathbf{C}_n[z]$

Perturbation :
Neighborhood of polynomial p

$$N_\varepsilon(p) = \{\widehat{p} \in \mathbf{C}_n[z] : \|p - \widehat{p}\| \leqslant \varepsilon\}\,.$$

Definition of the ε-pseudozero set :

$$Z_\varepsilon(p) = \{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\}\,.$$

$\|\cdot\|$ a norm on the vector of the coefficients of p

Pseudozero set : the set of the zeros of polynomials "near p".

Pseudozeros : brief survey of main references

- Mosier (1986) : Definition and study form the ∞-norm.
- Trefethen and Toh (1994) : Study for the 2-norm.
  pseudozeros $\approx$ pseudospectra of the companion matrix.
- Chatelin and Frayssé (1996) : propose a synthesis in *Lectures on Finite Precision Computations* (SIAM)
- Stetter (1999), (2004) : *Numerical Polynomial Algebra* (SIAM). Generalization of the previous works.
- Zhang (2001) : Study of the influence of the basis for the 2-norm (condition number of the evaluation).

Pseudozeros are easily computable

Theorem : (Mosier, Toh and Trefethen, Stetter)
The ε-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|\underline{z}\|_*} \leqslant \varepsilon \right\},$$

where $\underline{z} = (1, z, \ldots, z^n)$ and $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$,

$$\|y\|_* = \sup_{x \neq 0} \frac{|y^* x|}{\|x\|}$$

Algorithm of computation

Algorithm to draw the ε-pseudozero set :

1. We mesh a square containing all the roots of p (MATLAB command : `meshgrid`).
2. We compute $g(z) := \frac{|p(z)|}{\|\underline{z}\|_*}$ for all the nodes z in the grid.
3. We draw the contour level $|g(z)| = \varepsilon$ (MATLAB command : `contour`).

Problems :

- Find a square containing all the roots of p and all the pseudozeros.
- Find a grid step that separates all the roots.

A famous example

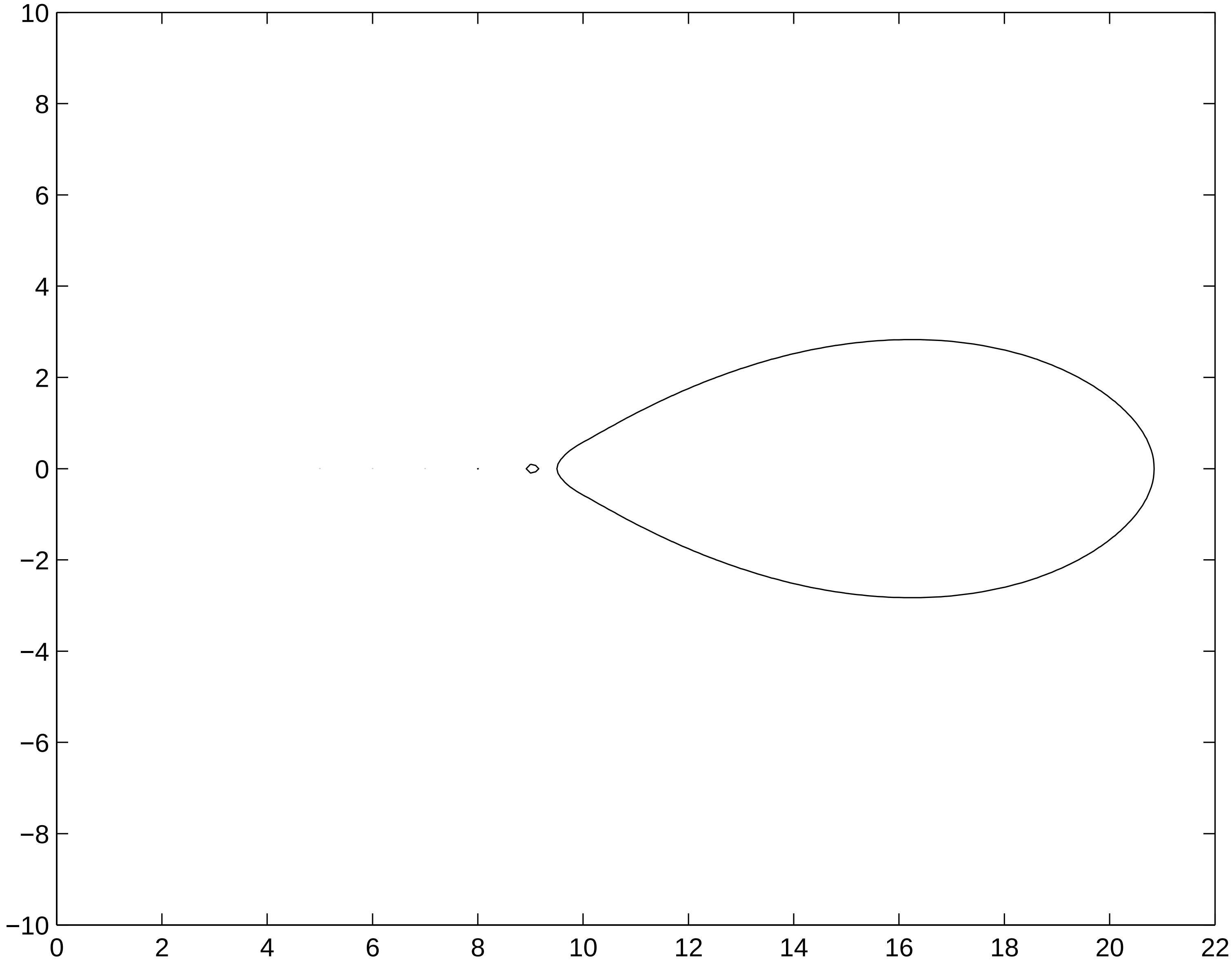
Pseudozero set of the *Wilkinson* polynomial

$$\begin{aligned} W_{20} &= (z-1)(z-2)\cdots(z-20), \\ &= z^{20} - 210z^{19} + \cdots + 20!. \end{aligned}$$

We perturb only the coefficient of z^{19} with $\varepsilon = 2^{-23}$.
One uses the weighted-norm $\|\cdot\|_\infty$:

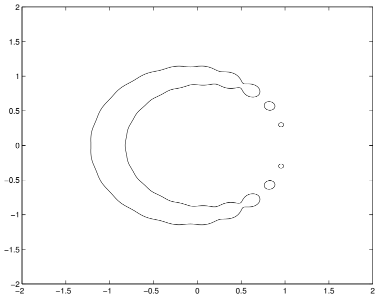
$$\|p\|_\infty = \max_i \frac{|p_i|}{m_i} \text{ with } m_i \text{ non negative}$$

with $m_{19} = 1$, $m_i = 0$ otherwise and the convention $m/0 = \infty$ if $m > 0$ and $0/0 = 0$.

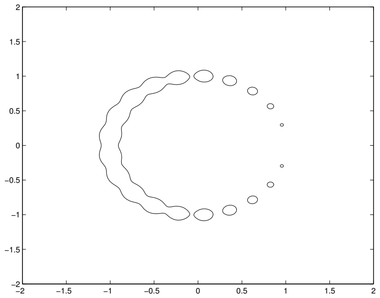

10
8
6
4
2
0
−2
−4
−6
−8
−10
0
2
4
6
8
10
12
14
16
18
20
22

Evolution of ε-pseudozero w.r.t. ε

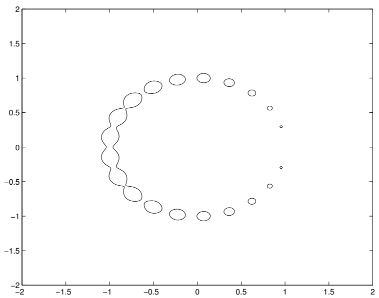
Pseudozero set of the polynomial $p(z) = 1 + z + \cdots + z^{20}$ for different values of ε.



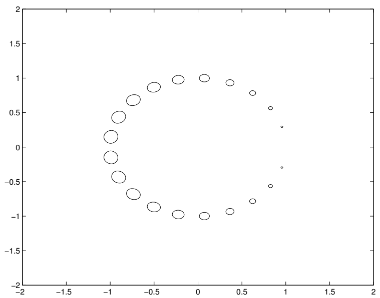
(a) $\varepsilon = 10^{-1}$



(b) $\varepsilon = 10^{-1.2}$



(c) $\varepsilon = 10^{-1.3}$



(d) $\varepsilon = 10^{-1.4}$

Application of pseudozeros to primality

Definition of ε-GCD of polynomials

Let p and q be two polynomials of degree n and m and let ε be a nonnegative number. We define

- an **ε-divisor** : a divisor of perturbed polynomials $\widehat{p}$ and $\widehat{q}$ satisfying
$$\deg \widehat{p} \leqslant n, \deg \widehat{q} \leqslant m \text{ and } \max(\|p - \widehat{p}\|, \|q - \widehat{q}\|) \leqslant \varepsilon.$$
- an **ε-GCD** : an ε-divisor of maximal degree.
- Two polynomials p and q are **ε-coprime** if their **ε-GCD** equals 1.

Definition of ε-primality

Remarks :

- ε measures the uncertainty about the coefficients (representing finite precision).
- Uniqueness of the degree but not of the ε-GCD.
- Dependency with respect to the basis field.

Computation :

- Optimization : algorithm of Karmarkar and Lakshman (1995).
- Sylvester criterion : algorithm COPRIME [Beckermann and Labahn 1998].
- Zeng (2004) : algorithm based on Gauss-Newton on a perjorative manifold
- Graphical : pseudozero set.

Pseudozeros to solve the ε-primality problem

From the definition of the ε-pseudozero set, we derive that

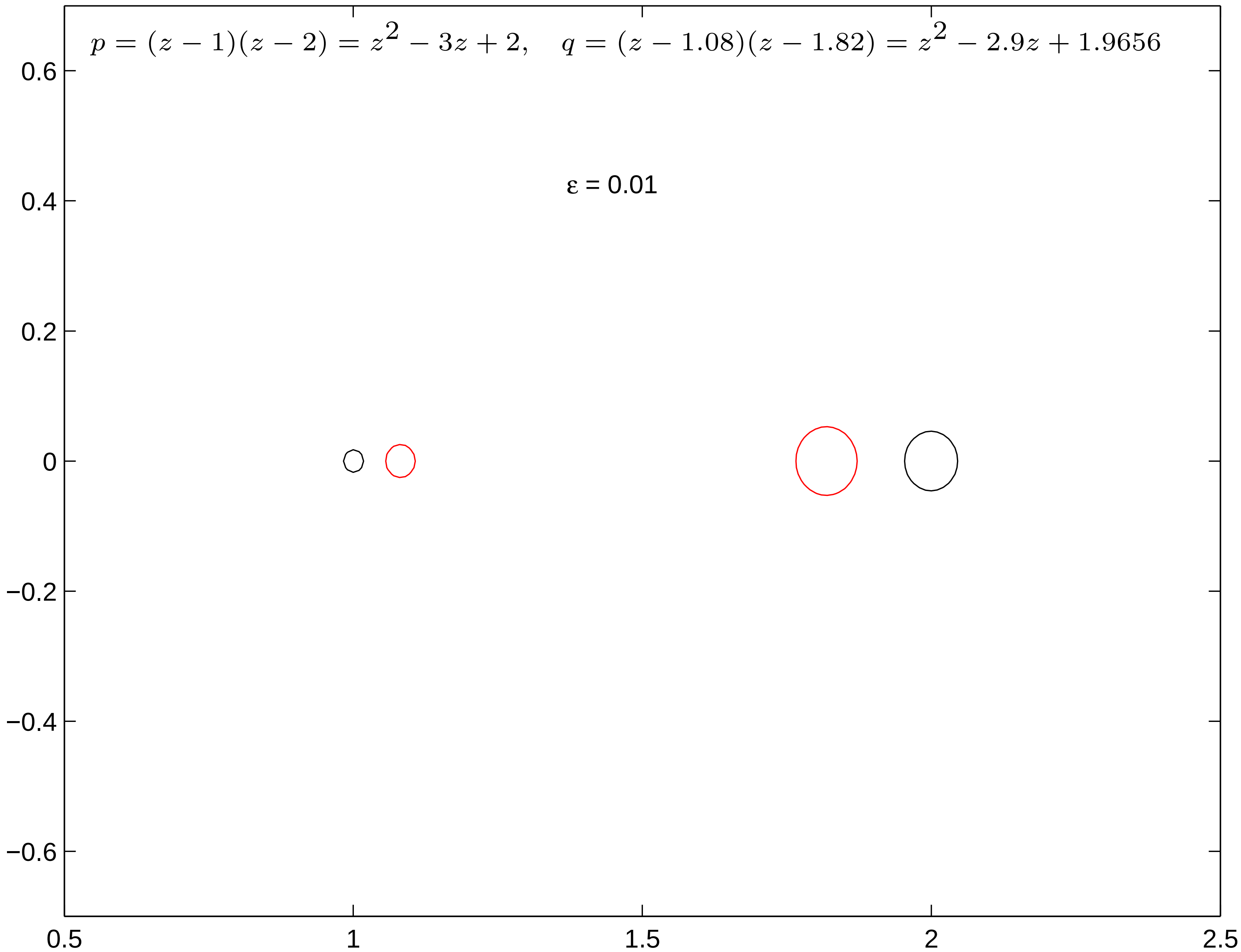
- if the intersection of the ε-pseudozero sets of p and q is empty then the two polynomials are ε-coprime,
- if the intersection is not empty then they are not ε-coprime.

Numerical simulation

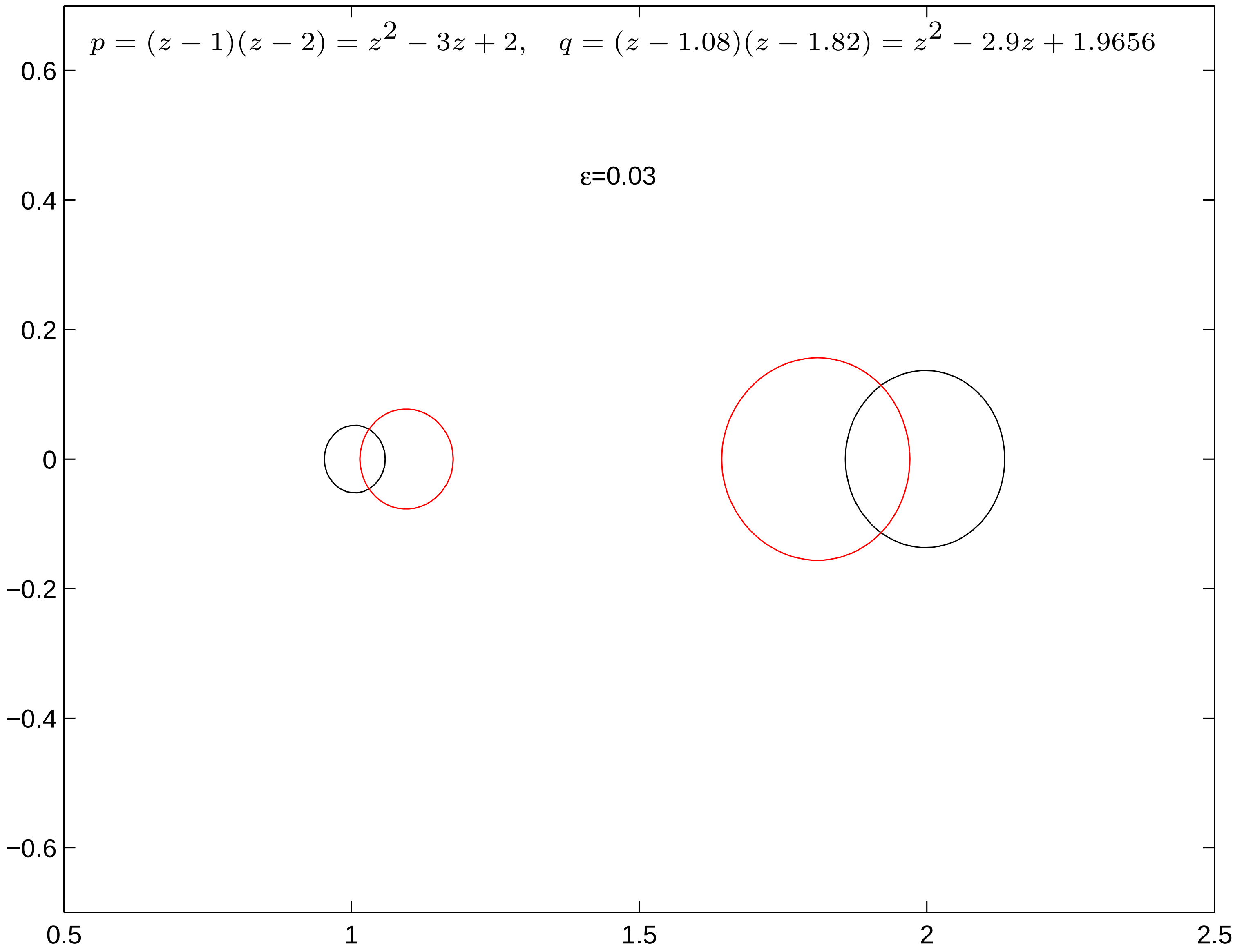
- **Input :** p and q two polynomials.
- **Output :** a graphic.
- **Drawbacks :** qualitative tool.
- **Example in $\|\cdot\|_2$:**

  $p = (z-1)(z-2) = z^2 - 3z + 2$

  $q = (z-1.08)(z-1.82) = z^2 - 2.9z + 1.9656$



$p = (z-1)(z-2) = z^2 - 3z + 2, \quad q = (z-1.08)(z-1.82) = z^2 - 2.9z + 1.9656$
ε = 0.01
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
0.5
1
1.5
2
2.5

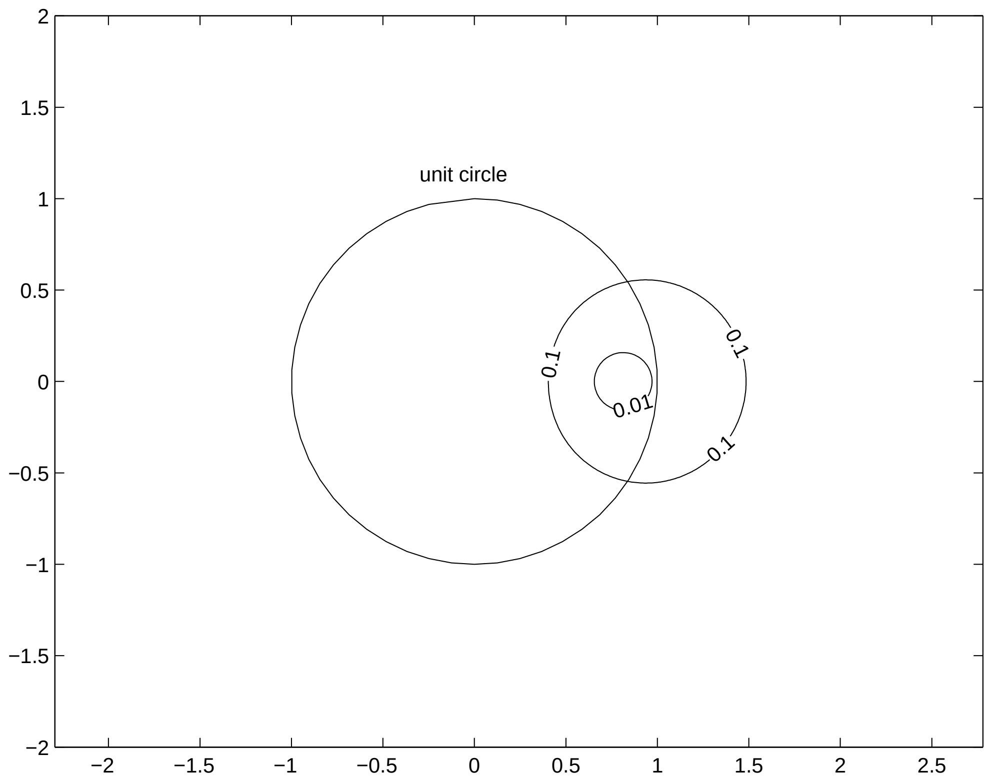

$p = (z-1)(z-2) = z^2 - 3z + 2, \quad q = (z-1.08)(z-1.82) = z^2 - 2.9z + 1.9656$
ε=0.03
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
0.5
1
1.5
2
2.5

Other applications of pseudozeros

Schur robust stability in control theory

Schur stability : $|\text{roots of } p| < 1$.

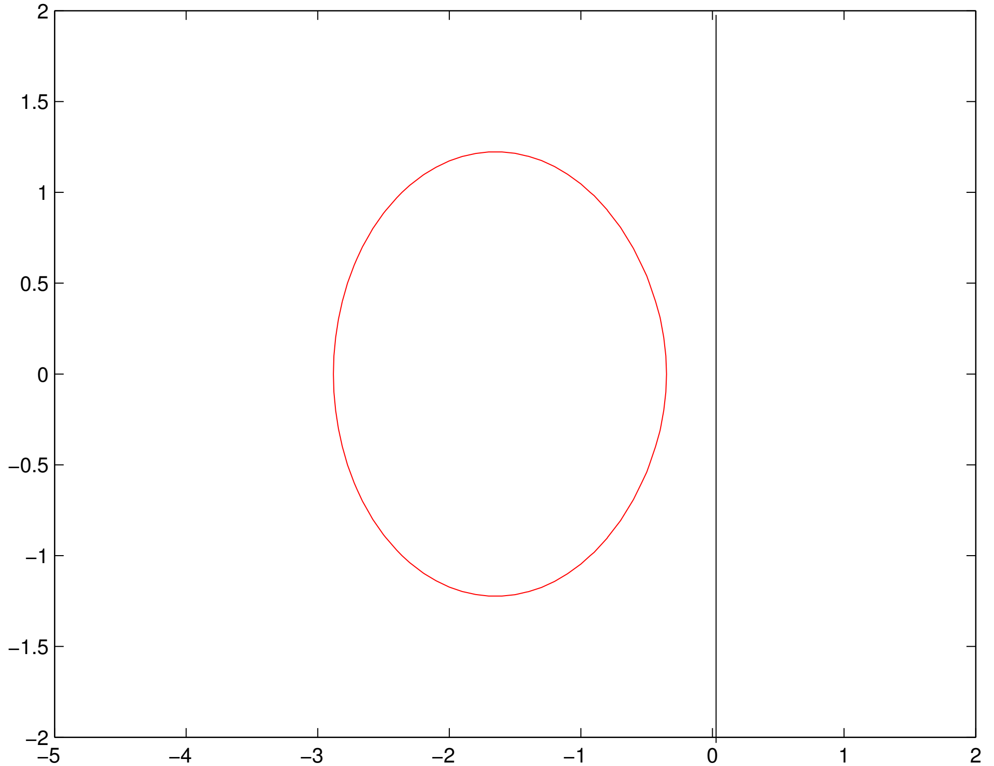
ε-pseudozero set of $p(z) = (z - 0.8)^2$ for $\varepsilon = 0.1$ and $\varepsilon = 0.01$.



Hurwitz robust stability in control theory

Hurwitz stability : Real part of roots of $p < 0$.

ε-pseudozero set of $p(z) = (z+1)^2$ for $\varepsilon = 0.4$.



Computation of stability radius

$\mathcal{P}_n$: polynomials of $\mathbf{C}[X]$ of degree less or equal than n
$\mathcal{M}_n$: monic polynomials of $\mathcal{P}_n$
$\|\cdot\|$: the 2-norm of the coefficients of a polynomial

Definition. *A polynomial is said to be* stable *if all the roots have negative real part and* unstable *otherwise (Hurwitz stability).*

The function *abscissa* $a : \mathcal{P} \to \mathbf{R}$ is defined by $a(p) = \max\{\mathrm{Re}(z) : p(z) = 0\}$.

A polynomial p is stable $\iff a(p) < 0$

Stability radius $\beta(p)$: distance of the polynomial $p \in \mathcal{M}_n$ from the set of monic unstable polynomials.

$$\beta(p) = \min\{\|p - q\| : q \in \mathcal{M}_n \text{ and } a(q) \geqslant 0\}.$$

Another characterization of $Z_\varepsilon(p)$

Let us denote $h_{p,\varepsilon} : \mathbf{R}^2 \to \mathbf{R}$ the function defined by

$$h_{p,\varepsilon}(x,y) = |p(x+iy)|^2 - \varepsilon^2 \sum_{j=0}^{n-1} (x^2+y^2)^j.$$

Then one has

$$Z_\varepsilon(p) = \{(x,y) \in \mathbf{R}^2 : h_{p,\varepsilon}(x,y) \leqslant 0\}$$

$\Longrightarrow$ $h_\varepsilon(\cdot, y)$ et $h_\varepsilon(x, \cdot)$ are polynomials of degree $2n$.

Theorem. *The equation $h_{p,\varepsilon}(0,y) = 0$ has a real solution y if and only if $\beta(p) \leqslant \varepsilon$.*

Algorithm (bisection)

Require : a stable polynomial p and a tolerance τ

Ensure : a number α such that $|\alpha - \beta(p)| \leqslant \tau$

$\gamma := 0, \quad \delta := \|p - z^n\|$

while $|\gamma - \delta| > \tau$ **do**

  $\varepsilon := \frac{\gamma+\delta}{2}$

  if the equation $h_{p,\varepsilon}(0, y) = 0, \;\; y \in \mathbf{R}$ has a solution **then**

    $\delta := \varepsilon$

  else

    $\gamma := \varepsilon$

  end if

end while

return $\alpha = \frac{\gamma+\delta}{2}$

Numerical simulation

For the polynomial $p(z) = z^2 + z + 1/2$, the algorithm gives $\beta(p) \approx 0.485868$ with $\tau = 0.00001$

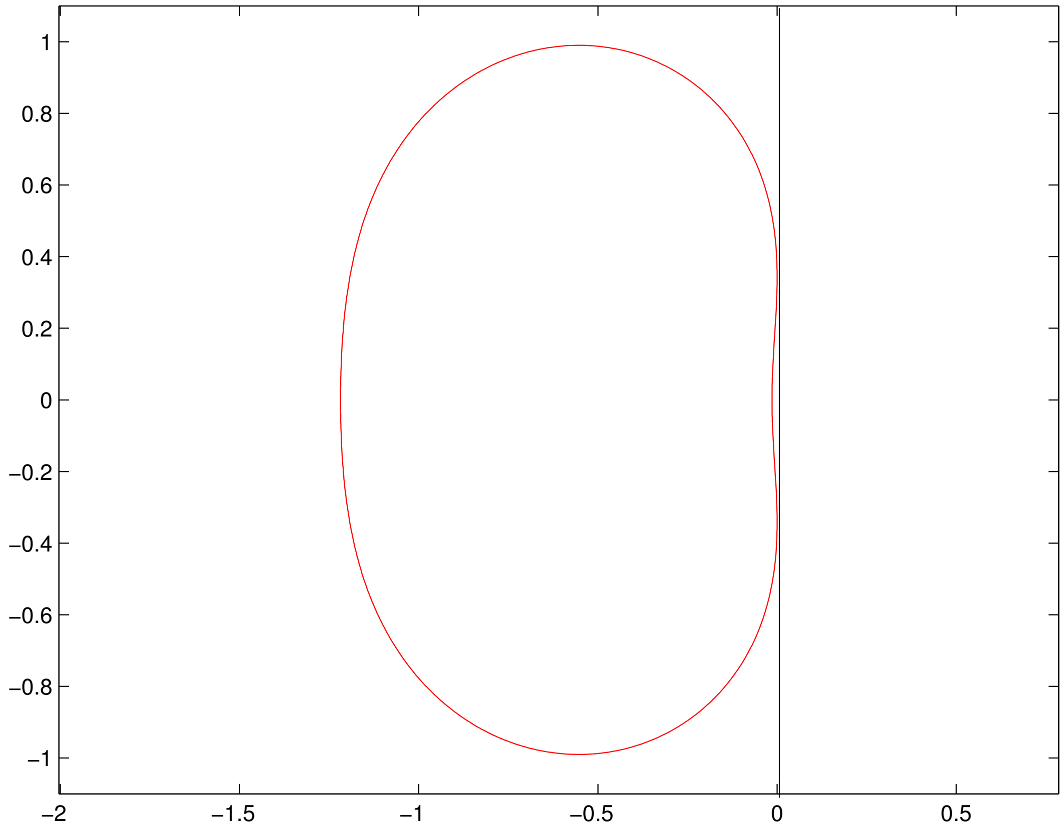


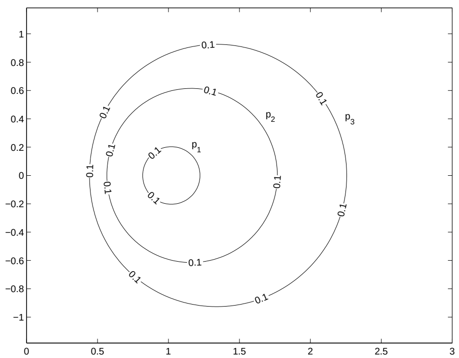
FIG. 1: $\beta(p)$-pseudozero set of $p(z) = z^2 + z + 1/2$

Multiplicity of polynomial roots

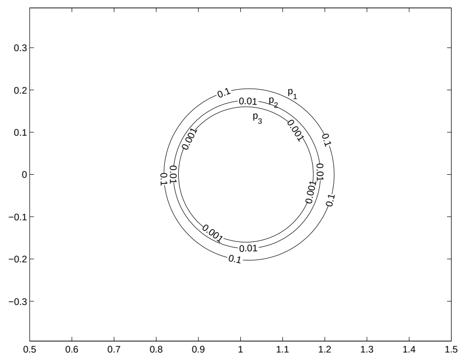
Computation of the ε-pseudozeros of polynomials :

$$p_1(z) = z - 1, \quad p_2(z) = (z-1)^2, \quad p_3(z) = (z-1)^3,$$

with, respectively, $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^2$, $\varepsilon_3 = \varepsilon^3$ and $\varepsilon = 10^{-1}$.



(a) Z_ε of p_1, p_2, p_3 and $\varepsilon = 10^{-1}$



(b) Pseudozero sets $Z_\varepsilon(p_1)$, $Z_{\varepsilon^2}(p_2)$, $Z_{\varepsilon^3}(p_3)$ for $\varepsilon = 10^{-1}$

Conclusion

The pseudozero set provides

1. a better understanding of the effect of coefficients perturbation ;
2. a test for ε-primality of two polynomials ;
3. an application for robust stability and multiplicity.