

# Some Topological And Geometric Properties Of Pseudozero Set\*

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## Abstract

The pseudozero set of a polynomial  $p$  is the set of complex numbers that are roots of polynomials which are near to  $p$ . This is a powerful tool to analyze the sensitivity of roots with respect to perturbations of the coefficients. Some applications in algebraic computation and robust control theory have been proposed recently. In this paper, we establish some topological and geometric properties of the pseudozero set such as boundedness, compactness and convexity.

## 1 Introduction

The computation of polynomial roots is extensively used in several fields of Scientific Computing and Engineering. The use of computers implies a round-off of the polynomial coefficients, often due to finite precision (in general using the IEEE 754 standard). The sensitivity of the roots with respect to the uncertainty of the polynomial coefficients has been studied with two main tools.

The first tool is concerned with the introduction of a *condition number* that estimates the magnitudes of the changes of the roots with respect to the changes of coefficients. A lot of work has been done in this direction mainly by Gautschi [7] and Wilkinson [22].

The idea of the second tool is to consider the uncertainty of the coefficients (due to round-off) as a continuity problem. This method was first introduced by Ostrowski [17]. The most powerful tool of this method seems to be the *pseudozero set* of a polynomial. Roughly speaking, it is the set of roots of polynomials which are near to a given polynomial. The pseudozero set was first introduced by Mosier [16] in 1986. He studied this set considering perturbations bounded with the  $\infty$ -norm. Trefethen and Toh [21] studied pseudozero set for perturbations bounded with the 2-norm. They also compared the pseudozero set of a given polynomial with the pseudospectra of the associated companion matrix (see also [6]). These results are summarized in Chatelin and Frayssé's book on finite precision [4]. More recently, Zhang [23] compared pseudozero set with respect to the choice of the polynomial basis (power, Taylor, Chebyshev, Bernstein). At last,

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recently, Stetter gave a general framework for working with inexact polynomials in his book [20]. The notion of root sets was introduced by Hinrichsen and Kelb [11]. It is a particular case of the spectral value sets of the companion matrix using structured perturbations. It corresponds exactly to the notion of pseudozero set but from a different viewpoint. Such a set was studied in particular by Hinrichsen and Kelb [11], Karow [14] and Hinrichsen and Pritchard [12].

Nevertheless, few applications of pseudozero set have been given in these previous publications, except when Bini and Fiorentino provided a multiprecision algorithm to compute polynomial root using pseudozero set [1]. Indeed, they need to know if an approximate root is a root of a nearby polynomial. Pseudozero set is the natural way to answer this question. Some applications of pseudozero set have been proposed recently in algebraic computation and robust control theory. An algorithm to test the  $\varepsilon$ -coprimeness of two numerical polynomials is proposed in [9]. Some applications in control theory (especially in robustness) have been proposed in [10] where the authors present an algorithm to compute the Hurwitz stability radius of a polynomial.

The major part of the papers cited above consider only the univariate case. The multivariate case seems to have received few attention. It has only been studied by Stetter in [20], by Hoffman, Madden and Zhang in [13] and Corless, Kai and Watt in [5]. Furthermore, the multivariate case has only been dealt with polynomials with complex coefficients. In [8], we consider systems where polynomials have real coefficients and such that all the polynomials in all the perturbed polynomial systems have real coefficients as well. We provide a simple criterion to compute the pseudozero set and study different methods to visualize it.

In this paper, we study the pseudozero set as a mathematical object. We derive some topological and geometric properties especially the convexity of the connected components of this set for small perturbations. The rest of the paper is organized as follows. In Section 2, we give some definitions and some known results about pseudozero set. In Section 3, we derive three topological and geometric properties of pseudozero set that are boundedness, compactness and convexity. Similar results for pseudospectra were given in Burke, Lewis, and Overton [2, 3]. In Section 4, we illustrate these properties by drawing examples of pseudozero sets. We conclude by giving some hints for future work.

## 2 Preliminaries

For  $n \geq 1$ , let  $\mathcal{P}_n$  be the linear space of polynomials of degree at most  $n$  with complex coefficients and  $\mathcal{M}_n$  be the subset of monic polynomials of degree  $n$ . Let  $p \in \mathcal{P}_n$  be a polynomial of degree  $n$  given by

$$p(z) = \sum_{k=0}^n p_k z^k.$$

Representing  $p$  by the vector  $(p_0, \dots, p_n)^T$  of its coefficients, we identify the norm  $\|\cdot\|$  on  $\mathcal{P}_n$  to the 2-norm on  $\mathbf{C}^{n+1}$  of the corresponding vector. Throughout the paper, we

will only work with the 2-norm denoted  $\|\cdot\|$ . It means that

$$\|p\| = \left( \sum_{k=0}^n |p_k|^2 \right)^{1/2}.$$

Given a real  $\varepsilon > 0$ , an  $\varepsilon$ -neighborhood of  $p$  is the set of all polynomials of  $\mathcal{P}_n$ , closed enough to  $p$ , that is,

$$N_\varepsilon(p) = \{\hat{p} \in \mathcal{P}_n : \|p - \hat{p}\| \leq \varepsilon\}.$$

The  $\varepsilon$ -pseudozero set of  $p$  is defined to include all the zeros of the  $\varepsilon$ -neighborhood of  $p$ . A definition of this set is

$$Z_\varepsilon(p) = \{z \in \mathbf{C} : \hat{p}(z) = 0 \text{ for some } \hat{p} \in N_\varepsilon(p)\}.$$

Theorem 1 below provides a computable counterpart of this definition.

**THEOREM 1.**(Trefethen and Toh [21]) The  $\varepsilon$ -pseudozero set of  $p$  verifies

$$Z_\varepsilon(p) = \left\{ z \in \mathbf{C} : g(z) := \frac{|p(z)|}{\|\underline{z}\|} \leq \varepsilon \right\},$$

where  $\underline{z} = (1, z, \dots, z^n)^T$ .

This theorem was proved in [21] for the 2-norm and in [19, 9] for an arbitrary norm. We recall the proof of [21] for completeness of the paper.

**PROOF.** If  $z \in Z_\varepsilon(p)$  then there exists  $\hat{p} \in \mathcal{P}_n$  such that  $\hat{p}(z) = 0$  and  $\|p - \hat{p}\| \leq \varepsilon$ . From Hölder's inequality  $|x^T y| \leq \|x\| \|y\|$ , we get

$$|p(z)| = |p(z) - \hat{p}(z)| = \left| \sum_{k=0}^n (p_k - \hat{p}_k) z^k \right| \leq \|p - \hat{p}\| \|\underline{z}\|.$$

It follows that  $|p(z)| \leq \varepsilon \|\underline{z}\|$ .

Conversely, let  $u \in \mathbf{C}$  be such that  $|p(u)| \leq \varepsilon \|\underline{u}\|$ . If  $u \neq 0$ , we can write  $u = |u|e^{i\theta}$ ,  $\theta \in [0, 2\pi)$  with  $|u| > 0$ . Let us introduce the polynomials  $r$  and  $p_u$  defined by

$$\begin{aligned} r(z) &= \sum_{k=0}^n r_k z^k \quad \text{with} \quad r_k = |u|^k e^{-ik\theta}, \\ p_u(z) &= p(z) - \frac{p(u)}{r(u)} r(z). \end{aligned}$$

It is clear that  $r(u) = \|\underline{u}\|^2 = \|r\|^2$ , and  $p_u(u) = 0$ . So we have

$$\|p - p_u\| = \frac{|p(u)|}{|r(u)|} \|r\| \leq \varepsilon.$$

Hence we obtain that  $u \in Z_\varepsilon(p)$ .

If  $u = 0$ , let us define  $p_u(z) = p(z) - p(u)$ . It is clear that  $p_u(u) = 0$ . Besides, we have  $\|p - p_u\| = |p(u)| \leq \varepsilon$  by hypothesis. In the same way, we get that  $u \in Z_\varepsilon(p)$ .

This theorem gives us an efficient way to compute the pseudozero set. MATLAB provides primitives that allow us to plot pseudozeros with the following very simple Algorithm 1.

ALGORITHM 1. Computation of  $\varepsilon$ -pseudozero set (MATLAB version)

**Require:** polynomial  $p$  and precision  $\varepsilon$

**Ensure:** pseudozero set layout in the complex plane

- 1: We grid a square containing all the roots of  $p$  with the MATLAB command `meshgrid`.
- 2: We compute  $g(z)$  at the grid nodes  $z$ .
- 3: We draw the level line  $|g(z)| = \varepsilon$  with the MATLAB command `contour`.

The initial grid must satisfy the two following conditions:

- The zeros and pseudozeros are included in its range.
- The roots are isolated by the grid discretization.

We discuss how to fulfill the second condition. The first one is discussed later in Proposition 1.

We need a grid that ensures the roots of  $p$  are isolated (with respect to the grid cells). The discretization step of the grid must be chosen consequently. The following lower bound for the distance between two distinct zeros of a polynomial is proposed by Mignotte [15]. If  $\{z_j\}$  is the set of its distinct roots, we have

$$\min_{j \neq k} |z_j - z_k| > \sqrt{\frac{|\Delta|}{n^{n+2} \|p\|_2^{n-1}}} =: \gamma,$$

where  $|\Delta|$  denotes the discriminant of the polynomial. For the drawing of the pseudozero sets, we choose a grid with step  $\gamma$ .

### 3 Some Topological and Geometric Properties

In this section, we establish various topological and geometric properties of the pseudozero set. Under reasonable assumption on  $\varepsilon$ , we prove in the first subsection that the  $\varepsilon$ -pseudozero set is compact. The second subsection is devoted to a result on convexity for the pseudozero set: we prove that for sufficiently small  $\varepsilon$ , the connected components of the pseudozero set of a polynomial with simple zeros are convex.

#### 3.1 Two Compactness Results

The  $\varepsilon$ -pseudozero is not necessarily bounded. For example, we can choose  $p(z) = z$  in  $\mathcal{P}_1$ . In this case, we have

$$g(z) = \frac{|z|}{\sqrt{1 + |z|^2}}.$$

For  $z \in \mathbf{C}$ , we have  $g(z) < 1$ . So, for  $\varepsilon = 1$ , the  $\varepsilon$ -pseudozero set equals the complex plane.

Let  $p \in \mathcal{P}_n$  be a polynomial of degree  $n$ . The unboundedness of the pseudozero set appears if a polynomial in  $N_\varepsilon(p)$  has a degree less than  $n$ . This may be avoided by choosing  $0 < \varepsilon < |p_n|$  as proved in Proposition 1.

PROPOSITION 1. Assume that  $0 < \varepsilon < |p_n|$ . Then the  $\varepsilon$ -pseudozero set  $Z_\varepsilon(p)$  is a compact set contained in the ball of radius  $\frac{\|p\| + \varepsilon}{|p_n| - \varepsilon}$ .

PROOF. As the function  $g$  is continuous, the set  $Z_\varepsilon(p) = g^{-1}([0, \varepsilon])$  is closed. Let us now show that it is bounded. Let us denote by  $\{z_j\}_{j=1:n}$  the roots of the polynomial  $p$  counted with their multiplicity and  $r = \max_j |z_j|$ . It is well known (see [15]) that

$$r \leq \frac{\|p\|}{|p_n|}.$$

If  $z \in Z_\varepsilon(p)$  then there exists  $\hat{p} \in \mathcal{P}_n$  satisfying both  $\hat{p}(z) = 0$  and  $\|p - \hat{p}\| \leq \varepsilon$ . It follows that

$$|z| \leq \frac{\|\hat{p}\|}{|\hat{p}_n|}.$$

A calculation yields

$$|\|\hat{p}\| - \|p\|| \leq \|\hat{p} - p\| \leq \varepsilon.$$

Consequently, we have  $\|\hat{p}\| \leq \|p\| + \varepsilon$  and so  $|z| \leq \frac{\|p\| + \varepsilon}{|\hat{p}_n|}$ . Moreover, from  $\|p - \hat{p}\| \leq \varepsilon$ , we derive  $|\hat{p}_n| \geq |p_n| - \varepsilon$  ( $|p_n| - \varepsilon > 0$  by assumption). To conclude, we have

$$|z| \leq \frac{\|p\| + \varepsilon}{|p_n| - \varepsilon}.$$

Sometimes, we prefer working with monic polynomials. Given a monic polynomial  $p \in \mathcal{M}_n$ , we can modify the definition of the neighborhood by

$$N_\varepsilon^m(p) = \{\hat{p} \in \mathcal{M}_n : \|p - \hat{p}\| \leq \varepsilon\}.$$

The superscript  $m$  in  $N_\varepsilon^m(p)$  stands for ‘‘monic’’. Then the new  $\varepsilon$ -pseudozero set of  $p$  is defined to include all the zeros of the  $\varepsilon$ -neighborhood of  $p$ . A definition of this set is

$$Z_\varepsilon^m(p) = \{z \in \mathbf{C} : \hat{p}(z) = 0 \text{ for some } \hat{p} \in N_\varepsilon^m(p)\}.$$

Following Theorem 2 provides a computable counterpart of this new definition.

THEOREM 2. The  $\varepsilon$ -pseudozero set of  $p$  verifies

$$Z_\varepsilon^m(p) = \left\{ z \in \mathbf{C} : g(z) := \frac{|p(z)|}{\|\underline{z}\|} \leq \varepsilon \right\},$$

where  $\underline{z} = (1, z, \dots, z^{n-1})^T$ .

PROOF. The proof is the same one as for Theorem 1 except that we now define  $r(z) = \sum_{k=0}^{n-1} r_k z^k$  with  $r_k = |u|^k e^{-ik\theta}$  so that  $p_u$  is a monic polynomial.

We can prove that the  $\varepsilon$ -pseudozero set  $Z_\varepsilon^m(p)$  is always bounded.

PROPOSITION 2. For all  $\varepsilon > 0$ , the  $\varepsilon$ -pseudozero set  $Z_\varepsilon^m(p)$  is a compact set contained in the ball of radius  $\|p\| + \varepsilon$ .

PROOF. It is the same proof as for Proposition 1 taking into account that  $p_n = 1$  and there is no perturbation on this coefficient.

### 3.2 A Convexity Result

In this section, we shall prove that for  $p \in \mathcal{P}_n$  of degree  $n$  with simple roots and  $\varepsilon$  sufficiently small, the connected components of the  $\varepsilon$ -pseudozero set are convex. This property is proved by using abstract results on Hilbert spaces.

DEFINITION 1.(Stetter [19]) Each maximal connected subset of a pseudozero set is called a *pseudozero component*.

The following proposition proves that, if the pseudozero set is bounded, each pseudozero component contains at least one root of the polynomial.

PROPOSITION 3.(Mosier [16]) Given  $p \in \mathcal{P}_n$  of degree  $n$ , assume that the pseudozero set  $Z_\varepsilon(p)$  is bounded. If  $q \in N_\varepsilon(p)$ , then  $p$  and  $q$  have the same number of roots, counting multiplicities, in each connected component of  $Z_\varepsilon(p)$ . Furthermore, there is at least one root of  $p$  in each connected component of  $Z_\varepsilon(p)$ .

This is Theorem 2 from [16].

We have the following proposition.

PROPOSITION 4. Let  $p \in \mathcal{P}_n$  be a polynomial of degree  $n$  with simple roots. There exists a value  $\bar{\varepsilon} > 0$  such that for each  $0 < \varepsilon < \bar{\varepsilon}$ , the pseudozero set  $Z_\varepsilon(p)$  is bounded and can be decomposed into  $n$  pseudozero components.

PROOF. Thanks to Proposition 1, there exists a value  $\varepsilon_1 > 0$  such that for  $0 < \varepsilon < \varepsilon_1$  the pseudozero set  $Z_\varepsilon(p)$  is bounded. Let  $(z_j)_{j=1:n}$  be the  $n$  simple roots of  $p$ . Let us denote

$$\text{sep} := \min_{j \neq k} |z_j - z_k| > 0.$$

By continuity of the roots with respect to coefficients (see Ostrowski [17]), there exists  $\eta > 0$  such that for all  $q \in N_\eta(p)$  there exists an order on the roots such that  $|z'_j - z_j| \leq \text{sep}/3$  where  $(z'_j)_{j=1:n}$  are the  $n$  roots of  $q$ . Let us denote  $\bar{\varepsilon} = \min\{\eta, \varepsilon_1\}$ . It follows that for  $0 < \varepsilon < \bar{\varepsilon}$  the  $\varepsilon$ -pseudozero set  $Z_\varepsilon(p)$  is decomposed into  $n$  pseudozero components.

Indeed, from Proposition 3, there are at most  $n$  pseudozero components. Moreover, each pseudozero component contains at least one zero of  $p$ . If  $z$  is a root of  $p$ , let us denote  $Z_\varepsilon(p, z)$  the pseudozero component of  $Z_\varepsilon(p)$  containing the root  $z$ . Let us suppose there exist  $i \neq j$  such that  $Z_\varepsilon(p, z_i) \cap Z_\varepsilon(p, z_j) \neq \emptyset$  for  $\varepsilon < \bar{\varepsilon}$ . As a consequence, there exists  $z \in Z_\varepsilon(p, z_i) \cap Z_\varepsilon(p, z_j)$  and so  $q_i, q_j \in N_\varepsilon(p)$  such that  $q_i(z) = q_j(z) = 0$ . By continuity of the roots with respect to coefficients, it holds  $|z - z_i| \leq \text{sep}/3$  and  $|z - z_j| \leq \text{sep}/3$ . It means that  $|z_i - z_j| \leq |z_i - z| + |z - z_j| \leq 2\text{sep}/3 < \text{sep}$  which contradicts the definition of  $\text{sep}$ . As a consequence, there are at least  $n$  pseudozero components (the  $Z(p, z_i)$  for  $i = 1 : n$ ) and so there are exactly  $n$  pseudozero components.

The above Proposition 3 needs the pseudozero set to be bounded. This justifies the study in Proposition 1 that provides simple conditions to assert it.

Given  $p = \sum_{j=0}^n p_j z^j$  and  $q = \sum_{j=0}^n q_j z^j$ , we define the inner product

$$\langle \cdot, \cdot \rangle : \mathcal{P}_n \times \mathcal{P}_n \rightarrow \mathbf{R}$$

by

$$\langle p, q \rangle = \operatorname{Re} \left( \sum_{j=0}^n p_j \overline{q_j} \right).$$

The space  $\mathcal{P}_n$  endowed with the inner product  $\langle \cdot, \cdot \rangle$  is a real Hilbert space. Let us consider the holomorphic function  $G$  defined by

$$G : \begin{cases} \mathcal{P}_n \times \mathbf{C} & \longrightarrow \mathbf{C}, \\ (q, z) & \longmapsto q(z). \end{cases}$$

Let  $p$  in  $\mathcal{P}_n$  be a polynomial of degree  $n$  with simple roots and let  $z_1$  be one root of  $p$ . We have

$$\frac{\partial G}{\partial z}(p, z_1) = p'(z_1) \neq 0.$$

Using the implicit function theorem, we obtain that there exist an open neighborhood  $U_1$  of  $p$  in  $\mathcal{P}_n$ , an open neighborhood  $V_1$  of  $z_1$  in  $\mathbf{C}$  and a holomorphic function  $\varphi : U_1 \rightarrow V_1$  such that

$$\forall q \in U_1, \quad \forall z \in V_1, \quad z = \varphi(q) \iff q(z) = 0.$$

Since  $U_1$  is open, there exists  $\eta_1 > 0$  such that  $N_{\eta_1}(p) \subset U_1$ . The function  $\varphi$  being holomorphic, it is a smooth function of class  $\mathcal{C}^\infty$  and so we get that  $\varphi'$  is Lipschitz continuous on  $N_{\eta_1}(p)$ . Since  $\varphi'(p)$  is a  $\mathbf{C}$ -linear form non identically zero, it is a surjection.

Let us now recall a result from B.T. Polyak [18]. Let  $X, Y$  be two real Hilbert spaces, let  $f : X \rightarrow Y$  be a nonlinear map with Lipschitz derivative on a ball  $B(a, r) = \{x \in X : \|x - a\| \leq r\}$ , thus

$$\|f'(x) - f'(y)\| \leq L\|x - y\|, \quad \forall x, y \in B(a, r). \quad (1)$$

Suppose that

$$\text{the linear operator } f'(a) \text{ maps } X \text{ onto } Y. \quad (2)$$

Then we have the following theorem (from [18]).

**THEOREM 3.**(Polyak [18]) If (1) and (2) hold, then there exists  $\varepsilon_0 > 0$  such that if  $\varepsilon < \varepsilon_0$  then the image of the ball  $B(a, \varepsilon) = \{x \in X : \|x - a\| \leq \varepsilon\}$  under the map  $f$  is convex, *i.e.*  $F = \{f(x) : x \in B(a, \varepsilon)\}$  is a convex set in  $Y$ .

Thanks to Proposition 4, let  $\bar{\varepsilon} > 0$  such that for each  $0 < \varepsilon < \bar{\varepsilon}$ , the pseudozero set  $Z_\varepsilon(p)$  is bounded and can be decomposed into  $n$  pseudozero components.

Let us denote  $\varepsilon_1 = \min\{\eta_1, \bar{\varepsilon}, \varepsilon_0\}$  where  $\varepsilon_0$  is such that  $\varphi(N_\varepsilon(p))$  is convex for all  $0 < \varepsilon < \varepsilon_0$ . Let us denote  $Z_\varepsilon(p, z_1)$  the pseudozero component of  $Z_\varepsilon(p)$  containing the root  $z_1$  (it exists thanks to Proposition 3). From the definition of  $\varphi$ , it follows that for  $0 < \varepsilon < \varepsilon_1$ , we have  $Z_\varepsilon(p, z_1) = \varphi(N_\varepsilon(p))$ . This is due to connectedness. First, it is clear that  $\varphi(N_\varepsilon(p)) \subset Z_\varepsilon(p)$ . Moreover,  $N_\varepsilon(p)$  is connected and  $\varphi$  is continuous so  $\varphi(N_\varepsilon(p))$  is also connected. From the choice of  $\varepsilon_1$ ,  $Z_\varepsilon(p)$  is decomposed into  $n$  pseudozero components so that  $\varphi(N_\varepsilon(p))$  must be included in a connected component. Since  $z_1 \in \varphi(N_\varepsilon(p))$ , necessarily  $\varphi(N_\varepsilon(p)) \subset Z_\varepsilon(p, z_1)$ .

Let now  $z \in Z_\varepsilon(p, z_1)$ . As a consequence, there exists  $q \in N_\varepsilon(p)$  such that  $q(z) = 0$ . We want to prove that  $z = \varphi(q)$ . As  $\varphi(N_\varepsilon(p)) \subset Z_\varepsilon(p, z_1)$ , necessarily  $\varphi(q) \in Z_\varepsilon(p, z_1)$  and is a root of  $q$ . Since from Proposition 3,  $q$  has only one root in  $Z_\varepsilon(p, z_1)$  then  $z = \varphi(q)$  and so  $Z_\varepsilon(p, z_1) \subset \varphi(N_\varepsilon(p))$ . As a consequence  $Z_\varepsilon(p, z_1) = \varphi(N_\varepsilon(p))$  is convex thanks to Theorem 3.

We can do the same thing with the roots  $z_i, i = 2 : n$  of  $p$ . In this case, we obtain  $\varepsilon_i$ , such that for  $0 < \varepsilon < \varepsilon_i$ , the pseudozero component  $Z_\varepsilon(p, z_i)$  of the pseudozero set  $Z_\varepsilon(p)$  containing  $z_i$  is convex. Let us denote  $\varepsilon' = \min_{i=1:n} \{\varepsilon_i\}$ . It follows that for  $\varepsilon < \varepsilon'$  the pseudozero set  $Z_\varepsilon(p) = \cup_{i=1}^n Z_\varepsilon(p, z_i)$  is the union of  $n$  convex pseudozero components.

We now state the following theorem.

**THEOREM 4.** Let  $p \in \mathcal{P}_n$  be a polynomial of degree  $n$  with simple roots. Then there exists a small  $\varepsilon' > 0$  such that for all  $0 < \varepsilon < \varepsilon'$  the pseudozero set  $Z_\varepsilon(p)$  consists in the union of exactly  $n$  convex pseudozero components.

## 4 Some Illustrations of the Convexity of Pseudozero Sets

In this section, we give some drawings of pseudozero sets illustrating the convexity of the connected components of these sets for small perturbations (and for polynomials with simple roots).

The first example is the pseudozero set of  $p(z) = z^2 - (10.5 + 10.2i)z + 1.5 + 53.5i$  for two different values of  $\varepsilon$ . It is shown on Figure 1 that for  $\varepsilon = 0.0015$  the pseudozero set is composed of only one connected component which is not convex. For a smaller value of  $\varepsilon = 0.001$ , the pseudozero set is composed of two connected components which are convex.

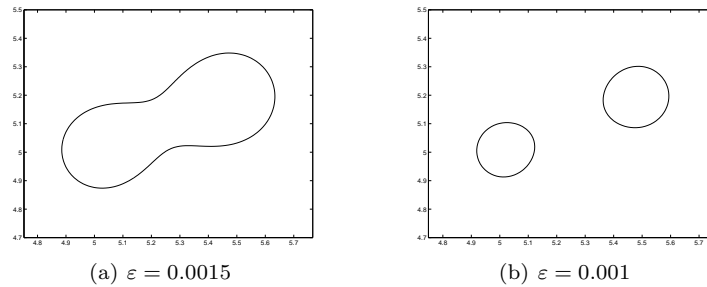


Figure 1: Pseudozero set of  $p(z) = z^2 - (10.5 + 10.2i)z + 1.5 + 53.5i$  for  $\varepsilon = 0.0015$  and  $\varepsilon = 0.001$

The second example is the pseudozero set of  $p(z) = 1 + z + z^2 + z^3 + \dots + z^{20}$  for two different values of  $\varepsilon$ . It is shown on Figure 2 that for  $\varepsilon = 0.5$  the pseudozero set is composed of five connected components. Only one of them is not convex. For a smaller



value of  $\varepsilon = 0.2$ , the pseudozero set is composed of twenty connected components which are all convex.

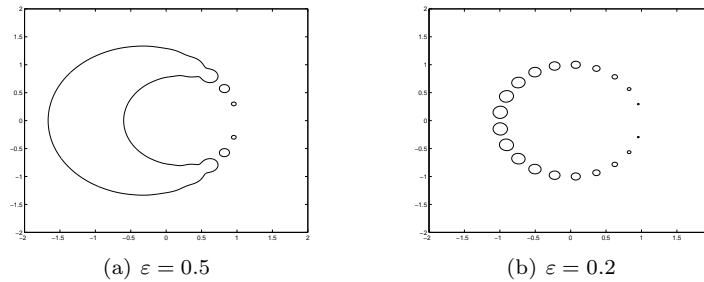


Figure 2: Pseudozero set of  $p(z) = 1 + z + z^2 + z^3 + \dots + z^{20}$  for  $\varepsilon = 0.5$  and  $\varepsilon = 0.2$

## 5 Conclusion and Future Work

In this paper, we have established some topological and geometric properties of the pseudozero set using the 2-norm. It seems clear that the properties of boundedness and compactness can be obtained for Hölder  $k$ -norm ( $1 \leq k \leq \infty$ ). But two questions appear naturally:

- is the convexity still true with the Hölder  $k$ -norm ( $1 \leq k \leq \infty$ )? Indeed, the technique used in this paper is not appropriate since the space  $\mathcal{P}_n$  endowed with  $\|\cdot\|_k$ ,  $k \neq 2$  is not a Hilbert space.
- is the convexity still true for polynomials having roots with multiplicities?

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