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Structured Condition Number and Backward Error for Eigenvalue Problems

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Abstract

In this paper, we investigate condition number and backward error for eigenvalue problems. Results on unstructured condition number for a simple eigenvalue are recalled and then a definition of a structured condition number is given for linear structures that are Toeplitz, circulant, Hankel, symmetric, Hermitian and skew-Hermitian. For these structures (except for circulant), we show that the unstructured condition number equals the structured condition number. We generalize these results to eigenvalues of matrix polynomials. We also study structured backward error for matrix polynomials.

Keywords: structured matrices, structured perturbations, matrix polynomials, condition number, backward error, eigenvalue problems, polynomial eigenvalue problems

Résumé

Dans ce papier, nous étudions le conditionnement et l'erreur inverse d'un problème de valeurs propres. Nous rappelons quelques résultats sur le conditionnement des valeurs propres simples. Nous définissons ensuite la notion de conditionnement structuré pour les structures Toeplitz, circulante, Hankel, symétrique, hermitienne et antihermitienne. Pour ces structures (excepté le cas circulant), nous montrons que le conditionnement structuré est égal au conditionnement non structuré. Nous généralisons ce résultats pour les problèmes de valeurs propres de matrices polynomiales. Enfin, nous étudions l'erreur inverse structurée pour le problème de valeurs propres de matrices polynomiales.

Mots-clés: matrices structurées, perturbations structurées, matrices polynomiales, nombre de conditionneent, erreur inverse, valeurs propres

Contents

1 Introduction and notation

Condition numbers play an important role in numerical linear algebra. They measure the sensitivity of the solution of a problem to perturbations in the data. Indeed, in practice, data are typically corrupted by errors. Three well known sources of approximation are considered in scientific computation [6]:

(1) errors due to discretization and truncation,

(2) errors due to roundoff, and

(3) errors due to uncertainty in the data.

Numerical methods for computing eigenvalues are mostly affected by rounding errors when working in finite precision. If the matrix has a given structure, it seems to be natural to take into account this structure. There are growing interests in algorithms for structured problems since few years (see, for example, [1, 4, 5] and the references therein). Then it is natural to define structured perturbation analysis, that is to say, to define new condition numbers with respect to structured perturbations.

Backward errors measure the stability of a numerical method. Using it with condition numbers, we can derive an upper bound for the error in a computed solution thanks to the well-known "rule of thumb",

$$\text{error} \lesssim \text{condition number} \times \text{backward error}.$$

This justifies the study of the backward error for the same problems as for condition numbers.

In this paper, we focus on the following linear structures,

$$\text{struct} \in \{\text{Toep}, \text{circ}, \text{Hankel}, \text{sym}, \text{Herm}, \text{skewHerm}\}, \tag{1.1}$$

corresponding to the set of Toeplitz, circulant, Hankel, symmetric, Hermitian and skew-Hermitian matrices, see Table 1. One will find in Table 2 the number of independent parameters for the structures we consider here.

Toeplitz matrices $(t_{i-j})_{i,j=0}^{n-1}$ Hankel matrices $(h_{i,j})_{i,j=0}^{n-1}$

$$\begin{pmatrix} t_0 & t_{-1} & \cdots & t_{1-n} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & t_1 & t_0 \end{pmatrix} \qquad \begin{pmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \iddots & h_n \\ \vdots & \iddots & \iddots & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n-2} \end{pmatrix}$$

Circulant matrices $(v_i)_{i=0}^{n-1}$

$$\begin{pmatrix} v_0 & v_{n-1} & \cdots & v_1 \\ v_1 & v_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_{n-1} \\ v_{n-1} & \cdots & v_1 & v_0 \end{pmatrix}$$

Table 1: Toeplitz, Hankel and circulant matrices

Structure	general	Toep	Hankel	circ	sym	Herm	skewHerm
k	n^2	$2n-1$	$2n-1$	n	$(n^2+n)/2$	$(n^2+n)/2$	$(n^2-n)/2$

Table 2: Number k of independent parameters

Throughout the paper, we denote by $M_n(\mathbf{C})$ the set of complex $n \times n$ matrices and $M_n^{\text{struct}}(\mathbf{C})$ the set of structured complex matrices, struct being defined in (1.1). We endow these spaces with the 2-norm (also called the spectral norm) denoted by $\|\cdot\|$. The superscript ^{T} denotes the transpose and * denotes the conjugate transpose. Throughout the paper, the matrix E is arbitrary and represents tolerances against which the perturbations are measured.

Let us consider a matrix $A \in M_n(\mathbf{C})$. Let λ be a *simple nonzero eigenvalue* of A with corresponding right eigenvector x and left eigenvector y so that $Ax = \lambda x$, and $y^*A = \lambda y^*$. We define the *condition number* of λ by

$$\begin{aligned}\kappa_E(A,\lambda) \quad = \quad \lim_{\varepsilon\to 0}\sup\Big\{ \frac{|\Delta\lambda|}{\varepsilon|\lambda|} &: (A+\Delta A)(x+\Delta x) = (\lambda+\Delta\lambda)(x+\Delta x), \\ &\Delta A \in M_n(\mathbf{C}), \quad \|\Delta A\| \le \varepsilon\|E\| \Big\}. \end{aligned} \tag{1.2}$$

It is well known [3, p.47] that

$$\kappa_E(A,\lambda) = \frac{\|E\|\|x\|\|y\|}{|y^*x||\lambda|}.$$

When the matrix A has a given structure, the entries are assumed to be defined according to this structure. This means that only perturbations on the entries are possible. For example, for a Toeplitz matrix, since only $2n-1$ coefficients define the matrix, we restrict only these $2n-1$ coefficients to be perturbed. This justifies the introduction of a structured condition number. Given a matrix $A \in M_n^{\text{struct}}(\mathbf{C})$, where struct belongs to (1.1), we define the *structured condition number* of λ by

$$\begin{aligned}\kappa_E^{\text{struct}}(A,\lambda) \quad = \quad \lim_{\varepsilon\to 0}\sup\Big\{ \frac{|\Delta\lambda|}{\varepsilon|\lambda|} &: (A+\Delta A)(x+\Delta x) = (\lambda+\Delta\lambda)(x+\Delta x), \\ &\Delta A \in M_n^{\text{struct}}(\mathbf{C}), \quad \|\Delta A\| \le \varepsilon\|E\| \Big\}. \end{aligned}$$

For $A \in M_n^{\text{struct}}(\mathbf{C})$, it is clear that we always have

$$\kappa_E^{\text{struct}}(A,\lambda) \le \kappa_E(A,\lambda).$$

We are interested in the structures for which $\kappa_E^{\text{struct}}(A,\lambda) = \kappa_E(A,\lambda)$.

The rest of the paper is organized as follows. In Section 2, we recall useful results. In Section 3, we prove that for Toeplitz, Hankel, symmetric, Hermitian and skew-Hermitian structures, the structured condition number equals the unstructured one. In Section 4, we generalize the previous result to the case of polynomial eigenvalue problems. In Section 5, we study the backward error for polynomial eigenvalue problems and we show that the structured backward error equals the unstructured backward error for Toeplitz, circulant, Hankel and symmetric structures.

2 Auxiliary results

In this section, we recall some known results. Rump proved the following result.

Lemma 2.1 (Rump [8, Lem. 10.1]). *Let $x \in \mathbf{C}^n$. Then there exists C, a complex Hankel matrix, such that $Cx = \overline{x}$ and $\|C\| = 1$.*

We will use the fact that

$$A \in M_n^{\text{Toep}}(\mathbf{C}) \Leftrightarrow AJ \in M_n^{\text{Hankel}}(\mathbf{C}) \Leftrightarrow JA \in M_n^{\text{Hankel}}(\mathbf{C}),$$

where J is the permutation matrix mapping $(1,\ldots,n)^T$ into $(n,\ldots,1)^T$,

$$J = \begin{pmatrix} (0) & & 1 \\ & \iddots & \\ 1 & & (0) \end{pmatrix}.$$

For a complex number z we define

$$\operatorname{sign}(z) = \begin{cases} \overline{z}/|z|, & z \ne 0, \\ 0, & z = 0. \end{cases}$$

For convenience, we define a function θ by

$$\theta(y,x) := \sup\{|y^*Fx| : F \in M_n(\mathbf{C}), \|F\| = 1\}.$$

It is clear that $\theta(y,x) \leq \|y\|\|x\|$. Defining $F = \frac{yx^*}{\|y\|\|x\|}$, we have

$$\|F\| = \frac{1}{\|y\|\|x\|} \max_{\|z\|=1} \|yx^*z\| = \frac{\|y\|}{\|y\|\|x\|} \max_{\|z\|=1} |x^*z| = 1,$$

and $|y^*Fx| = \|y\|\|x\|$ so that $\theta(y,x) = \|y\|\|x\|$. We define the structured version of this function by

$$\theta^{\text{struct}}(y,x) := \sup\{|y^*Fx| : F \in M_n^{\text{struct}}(\mathbf{C}), \|F\| = 1\}.$$

The following theorem exhibits the variation of a simple eigenvalue when a matrix is perturbed.

Theorem 2.2 (Stewart and Sun [9, p.183]). *Let λ be a simple eigenvalue of a matrix A, with right and left eigenvectors x and y, and let $\widehat{A} = A + \Delta A$ be a perturbation of A. Then there is a unique eigenvalue $\widehat{\lambda}$ such that*

$$\widehat{\lambda} = \lambda + \frac{y^*\Delta Ax}{y^*x} + \mathcal{O}(\|\Delta A\|^2).$$

Using Theorem 2.2 and the definition (1.2), it follows that (see [2])

$$\kappa_E(A,\lambda) = \frac{\|E\|}{|y^*x||\lambda|}\theta(y,x).$$

Using the same argument, it is easy to show that

$$\kappa_E^{\text{struct}}(A,\lambda) = \frac{\|E\|}{|y^*x||\lambda|}\theta^{\text{struct}}(y,x).$$

3 Structured condition number for eigenvalue problems

In this section, we prove that for struct $\in \{\text{Toep}, \text{Hankel}, \text{sym}, \text{Herm}, \text{skewHerm}\}$, the structured condition number equals the unstructured one.

3.1 Hermitian structure

Let $A \in M_n^{\text{Herm}}(\mathbf{C})$ be an Hermitian matrix with a simple eigenvalue λ (which is real). Let x be a right eigenvector of A. As A is Hermitian, it follows that x is also a left eigenvector. Then we have $\theta(x,x) = \|x\|^2$. We want to show that $\theta^{\text{struct}}(x,x) = \|x\|^2$ as well. It is clear that $\theta^{\text{struct}}(x,x) \leq \|x\|^2$. Let us consider $F = \frac{xx^*}{\|x\|^2}$. This matrix F is Hermitian and satisfies $\|F\| = 1$, $|x^*Fx| = \|x\|^2$. Then this implies that $\theta^{\text{Herm}}(x,x) = \theta(x,x)$ and so that $\kappa_E^{\text{Herm}}(A,\lambda) = \kappa_E(A,\lambda)$.

3.2 Skew-Hermitian structure

Let $A \in M_n^{\text{skewHerm}}(\mathbf{C})$ be a skew-Hermitian matrix with a simple eigenvalue $\lambda := i\mu$ with $\mu \in \mathbf{R}$. Let x be a right eigenvector of A. As A is skew-Hermitian, it follows that x is also a left eigenvector. Then we have $\theta(x,x) = \|x\|^2$. We want to show that $\theta^{\text{struct}}(x,x) = \|x\|^2$ as well. It is clear that $\theta^{\text{struct}}(x,x) \leq \|x\|^2$. Let us consider $F = \frac{ixx^*}{\|x\|^2}$. This matrix F is skew-Hermitian and satisfies $\|F\| = 1$, $|x^*Fx| = \|x\|^2$. Then this implies that $\theta^{\text{skewHerm}}(x,x) = \theta(x,x)$ and so that $\kappa_E^{\text{skewHerm}}(A,\lambda) = \kappa_E(A,\lambda)$.

3.3 Symmetric structure

Let $A \in M_n^{\text{sym}}(\mathbf{C})$ be a complex symmetric matrix with a simple eigenvalue λ. Let x be a right eigenvector of A. Since A is symmetric, it follows that $\overline{x}$ is a left eigenvector of A. It follows that we have $\theta(\overline{x},x) = \|x\|^2$. Let us show that $\theta^{\text{sym}}(\overline{x},x) = \|x\|^2$. It is clear that $\theta^{\text{sym}}(\overline{x},x) \leq \|x\|^2$. From Lemma 2.1, there exists a Hankel (so symmetric) matrix F such that $Fx = \overline{x}$ and $\|F\| = 1$. Then we have $|\overline{x}^*Fx| = \|\overline{x}\|^2 = \|x\|^2$. Then, it follows that $\theta^{\text{sym}}(\overline{x},x) = \theta(\overline{x},x)$ and so that $\kappa_E^{\text{sym}}(A,\lambda) = \kappa_E(A,\lambda)$.

3.4 Hankel structure

Let $A \in M_n^{\text{Hankel}}(\mathbf{C})$ be an Hankel matrix with a simple eigenvalue λ. Let x be a right eigenvector of A. Since A is symmetric (since Hankel), it follows that $\overline{x}$ is a left eigenvector of A. It follows that we have $\theta(\overline{x}, x) = \|x\|^2$. Let us show that $\theta^{\text{Hankel}}(\overline{x}, x) = \|x\|^2$. It is clear that $\theta^{\text{Hankel}}(\overline{x}, x) \le \|x\|^2$. From Lemma 2.1, there exists a Hankel matrix F such that $Fx = \overline{x}$ and $\|F\| = 1$. Then we have $|\overline{x}^* F x| = \|\overline{x}\|^2 = \|x\|^2$. Then, it follows that $\theta^{\text{Hankel}}(\overline{x}, x) = \theta(\overline{x}, x)$ and so that $\kappa_E^{\text{Hankel}}(A, \lambda) = \kappa_E(A, \lambda)$.

3.5 Toeplitz structure

Let $A \in M_n^{\text{Toep}}(\mathbf{C})$ be a Toeplitz matrix with a simple eigenvalue λ. Let x be a right eigenvector of A. Since A is Toeplitz, then JA is Hankel and so symmetric. As $Ax = \lambda x$, it follows that $JAx = \lambda Jx$. Let us denote $x_\sigma := Jx$. Then, it follows that $x^T JA = \lambda x_\sigma$, that is to say $x_\sigma^T A = \lambda x_\sigma^T$. It means that $\overline{x_\sigma}$ is a left eigenvector of A.

It follows that we have $\theta(\overline{x_\sigma}, x) = \|x\|^2$. Let us show that $\theta^{\text{Toep}}(\overline{x_\sigma}, x) = \|x\|^2$. It is clear that $\theta^{\text{Toep}}(\overline{x_\sigma}, x) \le \|x\|^2$. From Lemma 2.1, there exists a Hankel matrix F such that $Fx = \overline{x}$ and $\|F\| = 1$. Let us define the Toeplitz matrix $G := JF$. We have $\|G\| = 1$ and $Gx = \overline{x_\sigma}$. Then we have $|\overline{x_\sigma}^* G x| = \|x\|^2$. Then, it follows that $\theta^{\text{Toep}}(\overline{x_\sigma}, x) = \theta(\overline{x_\sigma}, x)$ and so that $\kappa_E^{\text{Toep}}(A, \lambda) = \kappa_E(A, \lambda)$.

4 Structured condition number for polynomial eigenvalue problems

The polynomial eigenvalue problem is to find the solutions $(x, \lambda) \in \mathbf{C}^n \times \mathbf{C}$ of

$$P(\lambda)x = 0, \tag{4.3}$$

where

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n(\mathbf{C})$, $k = 0 : m$. If $x \neq 0$ then λ is called an eigenvalue and x the corresponding eigenvector. We assume that P has only finite eigenvalues (and pseudoeigenvalues). Let us define

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n(\mathbf{C})$, $k = 0 : m$. We suppose now that λ is a nonzero simple eigenvalue with corresponding right eigenvector x and left eigenvector y (that is to say $P(\lambda)x = 0$ and $y^* P(\lambda) = 0$). Throughout the paper, the matrices E_k, $k = 0 : m$ allow freedom in how perturbations are measured. The condition number of λ can be defined by

$$\begin{aligned}\kappa_E(P, \lambda) \quad = \quad & \lim_{\varepsilon \to 0} \sup \Big\{ \frac{|\Delta\lambda|}{\varepsilon|\lambda|} : (P(\lambda + \Delta\lambda) + \Delta P(\lambda + \Delta\lambda))(x + \Delta x) = 0, \\ & \Delta A_k \in M_n(\mathbf{C}), \quad \|\Delta A_k\| \le \varepsilon \|E_k\|, k = 0 : m \Big\}.\end{aligned}$$

It is well known [11, Thm. 5] that

$$\kappa_E(P, \lambda) = \frac{\alpha \|y\| \|x\|}{|y^* P'(\lambda) x| |\lambda|}.$$

where $\alpha = \sum_{k=0}^m |\lambda|^k \|E_k\|$.

This result is a consequence of the following theorem that is presented in the proof of [11, Thm. 5].

Theorem 4.1. *Let λ be a simple eigenvalue of P, with right and left eigenvectors x and y, and let $\widehat{P}(\lambda) = P(\lambda) + \Delta P(\lambda)$ be a perturbation of P. Then there is a unique eigenvalue $\widehat{\lambda}$ such that*

$$\widehat{\lambda} = \lambda + \frac{y^* \Delta P(\lambda) x}{y^* P'(\lambda) x} + \mathcal{O}(\|\Delta P(\lambda)\|^2).$$

In fact, one can easily show with this theorem that

$$\kappa_E(P,\lambda) = \frac{\gamma(y,x)}{|y^*P'(\lambda)x||\lambda|},$$

where

$$\gamma(y,x) := \sup\{|y^*\Delta P(\lambda)x| : \Delta A_k \in M_n(\mathbf{C}), \|A_k\| \le \|E_k\|, k = 0:m\}.$$

It is shown in [11] that $\gamma(y,x) = \alpha\|y\|\|x\|$.

We assume now that the matrices ΔA_k have a certain structure belonging to

$$\text{struct} \in \{\text{Toep}, \text{Hankel}, \text{sym}, \text{Herm}, \text{skewHerm}\}. \tag{4.4}$$

We also suppose that all the matrices A_k and ΔA_k, $k = 0:n$, belong to $M_n^{\text{struct}}(\mathbf{C})$ for a given structure in (4.4). Let

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1}A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n^{\text{struct}}(\mathbf{C})$, $k = 0:m$ and

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1}\Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n^{\text{struct}}(\mathbf{C})$. One notices that $P(\lambda)$ and $\Delta P(\lambda)$ belong to $M_n^{\text{struct}}(\mathbf{C})$. The structured condition number of λ can be defined by

$$\begin{aligned}\kappa_E^{\text{struct}}(P,\lambda) \quad = \quad \lim_{\varepsilon\to 0}\sup\bigg\{&\frac{|\Delta\lambda|}{\varepsilon|\lambda|} : (P(\lambda+\Delta\lambda) + \Delta P(\lambda+\Delta\lambda))(x+\Delta x) = 0,\\ &\Delta A_k \in M_n^{\text{struct}}(\mathbf{C}), \quad \|\Delta A_k\| \le \varepsilon\|E_k\|, k = 0:m\bigg\}.\end{aligned}$$

As for the unstructured case, it is easy to show that

$$\kappa_E^{\text{struct}}(P,\lambda) = \frac{\gamma^{\text{struct}}(y,x)}{|y^*P'(\lambda)x||\lambda|},$$

where

$$\gamma^{\text{struct}}(y,x) := \sup\{|y^*\Delta P(\lambda)x| : \Delta A_k \in M_n^{\text{struct}}(\mathbf{C}), \|A_k\| \le \|E_k\|, k = 0:m\}.$$

It is clear from the definition of $\gamma^{\text{struct}}(y,x)$ that $\gamma^{\text{struct}}(y,x) \le \gamma(y,x)$.

4.1 Hermitian structure

Let $A_k \in M_n^{\text{Herm}}(\mathbf{C})$, $k = 0:m$ be some Hermitian matrices and λ a *real* simple eigenvalue of P. Let x be a right eigenvector of $P(\lambda)$. As A_k, $k = 0:m$ are Hermitian and λ is real, it follows that x is also a left eigenvector. It is shown in [11, Thm. 5] that $\gamma^{\text{Herm}}(x,x) = \gamma(x,x)$ and so that $\kappa_E^{\text{Herm}}(P,\lambda) = \kappa_E(P,\lambda)$.

4.2 Skew-Hermitian structure

Let $A_k \in M_n^{\text{skewHerm}}(\mathbf{C})$, $k = 0:m$ be some skew-Hermitian matrices and λ a *real* simple eigenvalue of P. Let x be a right eigenvector of $P(\lambda)$. As $P(\lambda)$ is skew-Hermitian, it follows that x is also a left eigenvector. Then we have $\gamma(x,x) = \alpha\|x\|^2$. We want to show that $\gamma^{\text{skewHerm}}(x,x) = \alpha\|x\|^2$ as well. It is clear that $\gamma^{\text{skewHerm}}(x,x) \le \alpha\|x\|^2$. Let us consider $F = \frac{ixx^*}{\|x\|^2}$. This matrix F is skew-Hermitian and satisfies $\|F\| = 1$, $|x^*Fx| = \|x\|^2$. Let

$$\Delta A_k = -\operatorname{sign}(\lambda^k)\|E_k\|F, \quad k = 0:m.$$

It follows that $\Delta A_k \in M_n^{\text{skewHerm}}(\mathbf{C})$ and $|x^*\Delta P(\lambda)x| = \alpha\|x\|^2$. Then this implies that $\gamma^{\text{skewHerm}}(x,x) = \gamma(x,x)$ and so that $\kappa_E^{\text{skewHerm}}(P,\lambda) = \kappa_E(P,\lambda)$.

4.3 Symmetric structure

Let $A_k \in M_n^{\rm sym}(\mathbf{C})$, $k = 0 : m$ be some complex symmetric matrices and λ a simple eigenvalue of P. Let x be a right eigenvector of $P(\lambda)$. Since $P(\lambda)$ is symmetric, it follows that $\overline{x}$ is a left eigenvector of $P(\lambda)$. It follows that we have $\gamma(\overline{x}, x) = \alpha\|x\|^2$. Let us show that $\gamma^{\rm sym}(\overline{x}, x) = \alpha\|x\|^2$. It is clear that $\gamma^{\rm sym}(\overline{x}, x) \leq \alpha\|x\|^2$. From Lemma 2.1, there exists a Hankel (so symmetric) matrix F such that $Fx = \overline{x}$ and $\|F\| = 1$. Then we have $|\overline{x}^* Fx| = \|\overline{x}\|^2 = \|x\|^2$. Let

$$\Delta A_k = -\,\mathrm{sign}(\lambda^k)\|E_k\|F, \quad k = 0 : m.$$

It follows that $\Delta A_k \in M_n^{\rm sym}(\mathbf{C})$ and $|\overline{x}^*\Delta P(\lambda)x| = \alpha\|x\|^2$ and so that $|\overline{x}^*\Delta P(\lambda)x| = \alpha\|\overline{x}\|\|x\|$. Then, it follows that $\gamma^{\rm sym}(\overline{x}, x) = \gamma(\overline{x}, x)$ and so that $\kappa_E^{\rm sym}(P, \lambda) = \kappa_E(P, \lambda)$.

4.4 Hankel structure

Let $A_k \in M_n^{\rm Hankel}(\mathbf{C})$, $k = 0 : m$ be some Hankel matrices and λ a simple eigenvalue of P. Let x be a right eigenvector of $P(\lambda)$. Since $P(\lambda)$ is symmetric (since Hankel), it follows that $\overline{x}$ is a left eigenvector of $P(\lambda)$. It follows that we have $\gamma(\overline{x}, x) = \alpha\|x\|^2$. Let us show that $\gamma^{\rm Hankel}(\overline{x}, x) = \alpha\|x\|^2$. It is clear that $\gamma^{\rm Hankel}(\overline{x}, x) \leq \alpha\|x\|^2$. From Lemma 2.1, there exists a Hankel matrix F such that $Fx = \overline{x}$ and $\|F\| = 1$. Then we have $|\overline{x}^* Fx| = \|\overline{x}\|^2 = \|x\|^2$. Let

$$\Delta A_k = -\,\mathrm{sign}(\lambda^k)\|E_k\|F, \quad k = 0 : m.$$

It follows that $\Delta A_k \in M_n^{\rm Hankel}(\mathbf{C})$ and $|\overline{x}^*\Delta P(\lambda)x| = \alpha\|x\|^2$. Then, it follows that $\gamma^{\rm Hankel}(\overline{x}, x) = \gamma(\overline{x}, x)$ and so that $\kappa_E^{\rm Hankel}(A, \lambda) = \kappa_E(A, \lambda)$.

4.5 Toeplitz structure

Let $A_k \in M_n^{\rm Toep}(\mathbf{C})$, $k = 0 : m$ be some Toeplitz matrices and λ a simple eigenvalue of P. Let x be a right eigenvector of $P(\lambda)$. Since $P(\lambda)$ is Toeplitz, then JA is Hankel and so symmetric. As $Ax = \lambda x$, it follows that $JP(\lambda)x = 0$. Let us denote $x_\sigma := Jx$. Then, it follows that $x^T JP(\lambda) = 0$, that is to say $x_\sigma^T P(\lambda) = 0$. It means that $\overline{x_\sigma}$ is a left eigenvector of $P(\lambda)$.

It follows that we have $\gamma(\overline{x_\sigma}, x) = \|x\|^2$. Let us show that $\gamma^{\rm Toep}(\overline{x_\sigma}, x) = \alpha\|x\|^2$. It is clear that $\gamma^{\rm Toep}(\overline{x_\sigma}, x) \leq \alpha\|x\|^2$. From Lemma 2.1, there exists a Hankel matrix F such that $Fx = \overline{x}$ and $\|F\| = 1$. Let us define the Toeplitz matrix $G := JF$. We have $\|G\| = 1$ and $Gx = \overline{x_\sigma}$. Then we have $|\overline{x_\sigma}^* Gx| = \|x\|^2$. Let

$$\Delta A_k = -\,\mathrm{sign}(\lambda^k)\|E_k\|F, \quad k = 0 : m.$$

It follows that $|\overline{x_\sigma}^*\Delta P(\lambda)x| = \alpha\|y\|\|x\|$. Then, it follows that $\gamma^{\rm Toep}(\overline{x_\sigma}, x) = \gamma(\overline{x_\sigma}, x)$ and so that $\kappa_E^{\rm Toep}(A, \lambda) = \kappa_E(A, \lambda)$.

5 Structured backward error for polynomial eigenvalue problems

The definition of the normwise backward error of an approximate eigenpair (x, λ) of (4.3) is

$$\eta(x, \lambda) := \min\{\varepsilon : (P(\lambda) + \Delta P(\lambda))x = 0,\ \|\Delta A_k\| \leq \varepsilon\|E_k\|,\ k = 0 : m\}$$

We recall some results from Tisseur [11].

Theorem 5.1 (Tisseur [11, Thm 1]). *The normwise backward error $\eta(x, \lambda)$ is given by*

$$\eta(x, \lambda) = \frac{\|r\|}{\alpha\|x\|}$$

where $r = P(\lambda)x$ and $\alpha = \sum_{k=0}^m |\lambda|^k\|E_k\|$.

When eigenvectors are not computed, a more appropriate measure of the backward error is

$$\eta(\lambda) := \min_{x\neq 0} \eta(x,\lambda).$$

Lemma 5.2 (Tisseur [11, Lem 3]). *If λ is not an eigenvalue of P then*

$$\eta(\lambda) = \frac{1}{\alpha\|P(\lambda)^{-1}\|},$$

where $\alpha = \sum_{k=0}^{m} |\lambda|^k \|E_k\|$.

The following lemma shows a relation between the backward error and the distance to singularity.

Lemma 5.3. *If λ is not an eigenvalue of P then*

$$\eta(\lambda) = \min\{\varepsilon : \det(P(\lambda)+\Delta P(\lambda)) = 0,\ \|\Delta A_k\| \le \varepsilon\|E_k\|,\ k = 0:m\}.$$

Proof. We have

$$\begin{aligned}
\eta(\lambda) &= \min_{x\neq 0} \eta(x,\lambda) \\
&= \min_{x\neq 0} \min\{\varepsilon : (P(\lambda)+\Delta P(\lambda))x = 0,\ \|\Delta A_k\| \le \varepsilon\|E_k\|,\ k = 0:m\} \\
&= \min\{\varepsilon : \det(P(\lambda)+\Delta P(\lambda)) = 0,\ \|\Delta A_k\| \le \varepsilon\|E_k\|,\ k = 0:m\}.
\end{aligned}$$

□

We consider now that the matrices ΔA_k have a certain structure belonging to (1.1). We also suppose that all the matrices A_k and ΔA_k, $k = 0:n$, belong to $M_n^{\text{struct}}(\mathbf{C})$ with struct $\in \{\text{Toep}, \text{circ}, \text{Hankel}, \text{sym}\}$. Let

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n^{\text{struct}}(\mathbf{C})$, $k = 0:m$ and

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n^{\text{struct}}(\mathbf{C})$. One notices that $P(\lambda)$ and $\Delta P(\lambda)$ belong to $M_n^{\text{struct}}(\mathbf{C})$.

The definition of the structured normwise backward error of an approximate eigenpair (x,λ) of (4.3) is

$$\eta^{\text{struct}}(x,\lambda) := \min\{\varepsilon : (P(\lambda)+\Delta P(\lambda))x = 0,\ \Delta A_k \in M_n^{\text{struct}}(\mathbf{C}),\ \|\Delta A_k\| \le \varepsilon\|E_k\|,\ k = 0:m\}$$

As for the unstructured case, we define

$$\eta^{\text{struct}}(\lambda) := \min_{x\neq 0} \eta^{\text{struct}}(x,\lambda).$$

The same proof as in Lemma 5.3 leads

$$\eta^{\text{struct}}(\lambda) = \min\{\varepsilon : \det(P(\lambda)+\Delta P(\lambda)) = 0,\ \Delta A_k \in M_n^{\text{struct}}(\mathbf{C}),\ \|\Delta A_k\| \le \varepsilon\|E_k\|,\ k = 0:m\}.$$

Let us recall some results from Rump [8]. Given a nonsingular matrix $A \in M_n(\mathbf{C})$, we define the distance to singularity by

$$d(A) = \min\{\|\Delta A\| : A+\Delta A \text{ singular}, \Delta A \in M_n(\mathbf{C})\}.$$

For a nonsingular matrix $A \in M_n^{\text{struct}}(\mathbf{C})$, we define the structured distance to singularity by

$$d^{\text{struct}}(A) = \min\{\|\Delta A\| : A+\Delta A \text{ singular}, \Delta A \in M_n^{\text{struct}}(\mathbf{C})\}.$$

Rump has proved in [8, Thm 12.2] that the two distances $d(A)$ and $d^{\text{struct}}(A)$ are equal for struct $\in \{\text{Toep}, \text{circ}, \text{Hankel}\}$.

Theorem 5.4 (Rump [8, Thm 12.2]). *Let nonsingular $A \in M_n^{\text{struct}}(A)$ be given for* struct *being Toeplitz, circulant or Hankel. Then we have*

$$d(A) = d^{\text{struct}}(A) = \|A^{-1}\|^{-1}.$$

The same property occurs for the symmetric structure. Before stating the result, we will need the two following lemmas.

Lemma 5.5 (Takagi's factorization). *If A is complex symmetric ($A^T = A$), then there exist a unitary matrix U and a real nonnegative diagonal matrix $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$ such that $A = U\Sigma U^T$.*

We refer to [7, Cor. 4.4.4] for a proof.

The next result can be found in Tisseur and Graillat [10]. For completeness, we recall the proof.

Theorem 5.6 (Tisseur and Graillat [10]). *Let nonsingular $A \in M_n^{\text{struct}}(\mathbf{C})$ be given for* struct *being symmetric. Then we have*

$$d(A) = d^{\text{struct}}(A) = \|A^{-1}\|^{-1} = \sigma_{\min}(A).$$

Proof. Obviously, we have $d^{\text{struct}}(A) \geq d(A) = \|A^{-1}\|^{-1} = \sigma_{\min}(A)$, and then it remains to show that $(A + \Delta A)x = 0$ for some $x \neq 0$ and ΔA symmetric with $\|\Delta A\| = \sigma_{\min}(A)$. Let $A = U\Sigma U^T$ be the Takagi's factorization of A where U is unitary and Σ is diagonal with nonnegative entries (see Lemma 5.5). Let x be the column of U corresponding to the smallest entry in Σ. Then $A\overline{x} = \sigma_{\min}(A)x$. By Lemma 2.1 there exists a symmetric matrix C such that $C\overline{x} = x$ and $\|C\| = 1$. Let $\Delta A = -\sigma_{\min}(A)C$. Then ΔA is symmetric, $\|\Delta A\| = \sigma_{\min}(A)$ and

$$(A + \Delta A)\overline{x} = \sigma_{\min}(A)x - \sigma_{\min}(A)x = 0$$

so that $A + \Delta A$ is singular. ☐

Theorem 5.7. *For* $\text{struct} \in \{\text{Toep}, \text{circ}, \text{Hankel}, \text{sym}\}$, *we have*

$$\eta^{\text{struct}}(\lambda) = \eta(\lambda) = \frac{1}{\alpha\|P(\lambda)^{-1}\|},$$

where $\alpha = \sum_{k=0}^{m} |\lambda|^k \|E_k\|$.

Proof. From the definition of the structured backward error, it is easy to see that we always have $\eta^{\text{struct}}(\lambda) \geq \eta(\lambda) = \frac{1}{\alpha\|P(\lambda)^{-1}\|}$. We just have to show that $\eta^{\text{struct}}(\lambda) \leq \frac{1}{\alpha\|P(\lambda)^{-1}\|}$. From Theorem 5.4, there exists $X \in M_n^{\text{struct}}(\mathbf{C})$ such that $P(\lambda) + X$ is nonsingular with $\|X\| = \|P(\lambda)^{-1}\|^{-1}$. Let ΔA_k be matrices defined by

$$\Delta A_k = \frac{1}{\alpha} \operatorname{sign}(\lambda^k) \|E_k\| X, \quad k = 0 : m,$$

where $\alpha = \sum_{k=0}^{m} |\lambda|^k \|E_k\|$. We have $\Delta P(\lambda) = X$ and moreover $\|X\| = \|P(\lambda)^{-1}\|^{-1} = \alpha\eta(\lambda)$. Then we deduce that $\eta^{\text{struct}}(\lambda) \leq \eta(\lambda)$ and so equality must hold. ☐

6 Conclusion

In this paper, we have shown that the structured condition number for a simple eigenvalue equals the unstructured condition number for the following structures: Toeplitz, Hankel, symmetric, Hermitian and skew-Hermitian. We have generalized these results for polynomial eigenvalue problems. Moreover, we have shown for the polynomial eigenvalue problem that the structured backward error equals the unstructured backward error for Toeplitz, Hankel, circulant and symmetric structures.

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