Fourier Analysis

Reminder:

\[\cos(p) + \cos(q) = 2\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)\]
\[\cos(p) - \cos(q) = -2\sin\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)\]
\[\sin(p) + \sin(q) = 2\sin\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)\]
\[\sin(p) - \sin(q) = 2\sin\left(\frac{p-q}{2}\right)\cos\left(\frac{p+q}{2}\right)\]

\[\cos(x) = \frac{e^{ix} + e^{-ix}}{2}\]
\[\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}\]
\[e^{inx} = \cos(nx) + i\sin(nx)\]

\[X(f) : f \mapsto \int_{-\infty}^{+\infty} x(t) e^{-2\pi i ft} dt\]

Exercise 1: Fourier Series

1. Let \(E\) be the vector space of continuous functions defined on \([-\pi, \pi]\) valued in \(\mathbb{C}\), with associated scalar product:

\[\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \bar{g}(t) dt\]

- Prove that the family \(\mathcal{F}\) defined by \(\{\cos(nx) \mid n \in \mathbb{N}\} \cup \{\sin(nx) \mid n \in \mathbb{N}^*\}\) is orthogonal.
- Is \(\mathcal{F}\) a basis for \(E\), the vector space of 2\(\pi\) periodic functions?
- Propose an orthonormal basis for \(E\).

2. Fourier Series: all 2\(\pi\) periodic function writes:

\[f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{+\infty} a_n \cos(nx) + b_n \sin(nx)\]

\[= \sum_{n \in \mathbb{Z}} c_n e^{inx}\]

Express coefficients \(c_n\) as function of \(a_n\) and \(b_n\).

3. Prove that the family \(\{e^{\frac{2\pi ikx}{T}}\}_{k \in \mathbb{Z}}\) is an orthogonal basis for the \(T\)-periodic functions of \(L^2([0,T])\).
Exercise 2: Fourier transform

1. Let \( z(t) = x(t - \tau) \), prove that \( Z(f) = e^{-2i\pi f \tau} X(f) \).

2. Prove that if \( x \) is pair (respectively impair), its Fourier transform is pair (respectively impair).

3. Let \( x \) be such as \( \lim_{t \to \pm \infty} x(t) = 0 \), let \( y = x' \), prove that \( Y(f) = 2i\pi f X(f) \).

4. Let \( y(t) = tx(t) \), prove that \( X'(f) = -2i\pi Y(f) \).

Exercise 3: Fourier transform of usual functions

1. Rect \((t)\), with \( \text{Rect}(t) = \begin{cases} 1 & \text{si } |t| \leq \frac{1}{2} \\ 0 & \text{sinon} \end{cases} \) (Gate or Rectangular function)

2. \( x(t) = e^{-\alpha |t|}, \alpha > 0 \)

3. \( g(t) = e^{-b^2 t^2}, \)
   - prove that \( g'(t) + 2b^2 tg(t) = 0 \)
   - deduce that \( G'(f) + \frac{2\pi}{b^2} G(f) = 0 \)
   - and that \( G(f) = \frac{\sqrt{\pi}}{|b|} e^{-\frac{\pi^2 f^2}{b^2}} \)

4. \( k(t) = e^{-\alpha t^2} \mathbb{1}_{t \geq 0}, \alpha > 0 \)

5. \( z(t) = t \mathbb{1}_{t \in [-a, a]} \)

Exercise 4: frequency resolution and windowing

Let us consider the Sine function \( x(t) = \cos(2\pi f_0 t) \) Rect \((t - \frac{T}{2}) \) and the Rectangular function \( r(t) = \text{Rect} \left( \frac{t}{\frac{T}{2}} \right) \). We recall that \( X(f) = \frac{1}{2} (\delta(f - f_0) + \delta(f + f_0)) \).

1. Determine the Fourier transform of \( z(t) = x(t)r(t) \).

2. What can we conclude about the frequency resolution?

Exercise 5: Short-time Fourier Transform

Reminder:

\[
\text{TFF}(x)(f, b) = \int_{\mathbb{R}} x(t) \bar{w}(t - b) e^{-2i\pi ft} dt
\]

Let’s consider the following 1-D signal:

\[
x(t) = \cos(2\pi f_1 t) \text{Rect} \left( \frac{t - T_1}{2T_1} \right) + \cos(4\pi f_1 t) \text{Rect} \left( \frac{t - 3T_1}{2T_1} \right)
\]

with \( f_1 = \frac{1}{T_1} \)

1. Draw the graph of signal \( x(t) \).

2. Determine and represent the spectrum of \( x \) for various disjoint temporal windows of length respectively \( \frac{1}{T} \) (1 window), \( \frac{2}{T} \) (2 windows), \( \frac{4}{T} \) (4 windows), \( \frac{1}{2T} \) (8 windows).

3. What previous windows can separate in time and in frequency the components of \( x \)? What is the best compromise?
Haar Wavelets

Let’s consider the vector space of square integrable functions defined on \( \mathbb{R} \) and denoted \( E = L^2(\mathbb{R}) \). Let \( \phi \) be the scale function defined on \( \mathbb{R} \) by:

\[
\phi(t) = \begin{cases} 
1 & t \in [0,1[ \\
0 & \text{otherwise}
\end{cases}
\]

and \( \phi_k^j(t) = \sqrt{2^j} \phi(2^j t - k) \):

\[
\phi_k^j(t) = \begin{cases} 
\sqrt{2^j} & t \in \left[ \frac{k}{2^j}, \frac{k+1}{2^j} \right[ \\
0 & \text{otherwise}
\end{cases}
\]

Exercise 6: multiresolution analysis of \( E \)

1. Prove that \( \phi \) is an admissible scale function to build a multiresolution analysis of \( E \).

2. Describe \( V^0, V^j \).

We recall the multiresolution analysis of a vector space \( E \):

1. \( \forall j \in \mathbb{Z} \quad V^j \subset V^{j+1} \)

2. \( \lim_{j \to -\infty} V^j = \bigcap_{j \in \mathbb{Z}} V^j = \emptyset \)

3. \( \lim_{j \to +\infty} V^j = \bigcup_{j \in \mathbb{Z}} V^j = E \)

4. \( \exists \phi \) such as \( \{ \phi(\cdot - n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( V^0 \)

5. \( \forall j \in \mathbb{Z}, f \in V^j \Leftrightarrow f(2^j \cdot) \in V^{j+1} \)

6. (consequence) \( \forall j, k \in \mathbb{Z}, f \in V^j \Leftrightarrow f(\cdot - 2^j k) \)

Exercise 7: orthonormal basis of Haar scale functions

Let’s consider the space of square integrable functions defined on \([0,1]\): \( E = L^2([0,1]) \). We apply a multiresolution analysis using \( \phi \), we have:

- \( V^0 \) the space of constant functions on \([0,1]\) (dimension 1)
- \( V^1 \) the space of constant functions on \([0,\frac{1}{2}]\) and \([\frac{1}{2},1]\) (dimension 2)
- \( V^j \) the space of constant functions on the \(2^j\) intervals \( \left[ \frac{k}{2^j}, \frac{k+1}{2^j} \right[ \), \( k = 0, \cdots, 2^j - 1 \) (dimension \( 2^j \))

1. Verify that the family of functions \( \left( \phi_k^j \right)_{k \in \{0,\cdots,2^j-1\}} \) is an orthonormal basis of \( V^j \).

2. Draw the graph of functions \( \phi_0^0 \) et \( \phi_1^1 \) (basis of \( V^1 \)), and of functions \( \phi_0^2, \phi_1^2, \phi_2^2, \phi_3^2 \) (basis of \( V^2 \)).
Exercise 8: orthonormal basis of Haar details functions

1. Determine the two elements \( \psi_0^1 \) and \( \psi_1^1 \) of the basis of the vector space \( W^1 \) such as \( V^2 = V^1 \oplus W^1 \).
   - Compression: express \( \phi_0^1, \phi_1^1, \psi_0^1 \) and \( \psi_1^1 \) as functions of \( (\phi_k^2)_{k \in \{0,1,2,3\}} \).
   - Uncompression: express \( (\phi_k^2)_{k \in \{0,1,2,3\}} \) as functions of \( \phi_0^1, \phi_1^1, \psi_0^1 \) and \( \psi_1^1 \).
   - Determine the function \( \psi \) such as \( \psi_{1}^{1}(t) = \sqrt{2}\psi(2t - k) \).

2. Generalization: deduce the definition of the \( 2^j \) Haar details functions \( (\psi_k^j)_{k \in \{0,\ldots,2j-1\}} \) as an orthonormal basis of spaces \( W^j \) such as \( V^{j+1} = V^j \oplus W^j \).
   a) Compression: express \( (\phi_k^j) \) and \( (\psi_k^j) \) as function of \( (\phi_k^{j+1}) \).
   b) Uncompression: express \( (\phi_k^{j+1}) \) as function of \( (\phi_k^j) \) and \( (\psi_k^j) \).
   c) Compression: express coefficients \( (s_k^j) \) et \( (d_k^j) \) as function of coefficients \( (s_k^{j+1}) \).
   d) Uncompression: express coefficients \( (s_k^{j+1}) \) as function of coefficients \( (s_k^j) \) and \( (d_k^j) \).
   e) Why the relation between coefficients \( s \) and \( d \) is the same than between functions \( \phi \) and \( \psi \)?

Exercise 9: representation of a signal in the Haar wavelet basis

Let \( S \) be the following discrete signal \[2, 4, 8, 12, 14, 0, 2, 1\].

1. Project \( S \) in the space \( V^0 \oplus W^0 \oplus W^1 \oplus W^2 \).

2. Draw the signal obtained for each resolution level (i.e. projection of \( S \) in \( V^2, V^1 \) and \( V^0 \)), and details coefficients (i.e. projection of \( S \) in \( W^2, W^1 \) and \( W^0 \)).

3. The module of the Discrete Fourier Transform of \( S \) is: \[9, 8, 11, 24, 44, 24, 11, 8\] (rounded to the closest integer).

Discuss the interpretation in terms of scale space of the signal in the Haar wavelet space and in the Fourier space.