TADI: Scale space theory
Master IMA/DIGIT
Sorbonne University

Dominique.Bereziat@lip6.fr
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Part 1: linear scale space
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Introduction

Earlier works: image decomposition
Continuous linear scale space: 1-D case
Continuous linear scale space in $\mathbb{R}^2$
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APPENDIX
In most of image processing problems:

**Axiom**

*Detection of an image primitive is possible only at an optimal scale*

**Axiom ([Marr and Hildreth, 1980])**

*An image is characterized by its local intensity variation at each scale*

- An image primitive: edges, contours, regions, pixels structures,
- Scale: an abstract measurement of image structures size, in practice, we measure in “pixels”
• Image resolution: number of pixels in an image
• and then: \( \text{scale} \neq \text{resolution} \)

• Range of scales are bounded:
  • minimal scale: 1 pixel, the real size may be known if we knows the pixel "size"
  • maximal scale: given by the image resolution (i.e. the number of pixels)

• Primitive ⇒ a local brightness variation ⇒ a differential operator to approximate ⇒ linear filter size suited to the image primitive size.
  Ex: \( f'(x) \approx f(x + 1) - f(x) \)
Importance in Image processing

- Optimization $\Rightarrow$ calculus of variation $\Rightarrow$ partial differential equations $\Rightarrow$ approximation of differential operators
- Continuous and discrete solutions belong to a scale space (admitted)
- A dedicated conference: *Conference on Scale Space and Variational Methods in Computer Vision*

Scope: Image analysis, Scale-space methods, Level sets methods, PDEs in image processing, Inverse problems in imaging, Compressed sensing, Optimization methods in imaging, Convex and non convex modeling and optimization in imaging, Restoration and reconstruction, Multi-scale shape analysis, 3D imaging modalities, 3D vision, Wavelets and image decomposition, Segmentation, Stereo reconstruction, Optical flow, Motion estimation, Registration, Surface modeling, Implicit surfaces, Shape from X, Inpainting, Color enhancement, Perceptual grouping, Selection of salient scales, Feature analysis, Cross-scale structure, Multi-Orientation Analysis, Differential geometry and Invariants, Mathematics of novel imaging methods, Sub-Riemannian geometry, Medical imaging.
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The idea is not novel:

- [Rosenfeld and Thurston, 1971]: bank of various operators with different sizes for edges detection.
- [Klinger, 1971, Uhr, 1972]: representation at various resolution (obtained by under-sampling) ⇒ pyramid of resolutions
  - consequence: primitive scales decreases with the image resolution
  - advantage: any primitive can be detected with the same operator: it exists an optimal representation (a level in the pyramid) to detect a given primitive
Pyramid of resolutions (1)

- Let's $K > 0$ an natural number, and $0 < k \leq K$
- $I^K \equiv I$ image with finest resolution
- $I^k$ image at resolution (level) $k$ is obtained from superior resolution $k + 1$:
  - applying an anti-aliasing filter (a Gaussian smoother)
  - and downsampling of factor 2 in the two image direction (image size divided by 4)
Pyramid of resolutions (2)

- Reduction operator $R$, in 1-D:

$$
f^{(k-1)}(x) = R(f^{(k)}) = \sum_{n \in \mathbb{N}} c(n)f^{(k)}(2x - n)
$$

with $c$ a low-pass filter

- In 2-D: we get a separable filter, $c(n, p) = c(n)c(p)$

$$
l^{(k-1)}(x, y) = R(f^{(k)}) = \sum_{n \in \mathbb{N}} \sum_{p \in \mathbb{N}} c(n, p)l^{(k)}(2x - n, 2y - p)
$$
Characterization of $c$ (1-D)

- **Spatial properties:**
  - finite support in $\mathbb{Z}$: $\exists N$ such as $\forall |n| > N$, $c(n) = 0$
  - positivity: $c(n) \geq 0$
  - unimodality: $c(|n|) \geq c(|n + 1|)$
  - symmetry: $c(n) = c(-n)$
  - normalization: $\sum c(n) = 1$
  - identical contribution at each level: $\sum c(2n) = \sum c(2n + 1)$

- **Frequency properties:**
  - low-pass filter: high frequencies components are set to zero to avoid aliasing
  - should approximate an ideal low-pass filter (ideal filter have an infinite support in frequency domain)

- **Examples:**
  - $N = 3$: unique filter: $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$
  - $N = 5$: $\left(\frac{1}{4} - \frac{a}{2}, \frac{1}{4}, a, \frac{1}{4}, \frac{1}{4} - \frac{a}{2}\right)$, with $a = 0.4$ we are closest to a Gaussian
Band-pass pyramid

- between each level, use a pass-band instead of low-pass
- construction: subtract adjacent levels of a low-pass pyramid,
  \[ L^{(k)} = f^{(k)} - \mathcal{E}(f^{(k-1)}) \]  
  \[ L^{(0)} = f^{(0)} \]  
- \( \mathcal{E} \) upsample operator to retrieve the previous resolution
- but: loss of information

\[ \mathcal{E}(f^{(k-1)}) = 2 \sum c(n)f^{(k-1)} \left( \frac{x-n}{2} \right) \]
Band-pass pyramid: applications

- Structure detection: image structures are localized in the pyramid levels compatible with their sizes
- Information is not redundant between the pyramid levels: low correlation between levels, each level may be efficiently compressed
- Easy to select some details levels
- Imperfect reconstruction is possible by inverting Eqs. (1, 2):

\[
\tilde{f}(0) = L^{(0)}
\]

\[
\tilde{f}(k) = L^{(k)} + \mathcal{E}(\tilde{f}^{(k-1)})
\]
Pyramid of resolutions: pros and cons

- **Pro:**
  - Easy to implement, low complexity
  - Allows to implement efficient multiresolution algorithms

- **Cons:**
  - non invariant by translation
  - non invariant by rotation

- To go further: [Burt, 1981], [Crowley, 1981]
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First definition of scale scale (in $\mathbb{R}$)

- Continuous case: scale parameter and spatial domain are continuous
- First definition appears in founding paper [Witkin, 1983] and addresses 1-D signals
- Given a signal $f$, we derive a family of signals $L$ indexed by a scale parameter $t \in \mathbb{R}^+$:

$$
\begin{align*}
  f : \mathbb{R} &\rightarrow \mathbb{R} \quad \text{original signal} \\
  L : \mathbb{R} \times \mathbb{R}^+ &\rightarrow \mathbb{R} \quad \text{scale representation}
\end{align*}
$$

- Definition of $L$:

$$
\begin{align*}
  L(x, t = 0) &= f(x) \\
  L(x, t) &= g_{\sigma(t)} * f(x), \quad t > 0
\end{align*}
$$
Choice for $g$

- $g$ is a convolution kernel
- a Gaussian of standard deviation $\sigma$:
  \[ g_\sigma = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \]

- Rem: $\sigma(t) = \sqrt{t}$
- Action on a signal:

![Image](image_url)

**Figure 1:** From [Witkin, 1983]
One more dimension

- \( L \) belongs to a space having one more dimension than those of \( f \)
- Supplementary dimension: a space of scale factor \( t \):

![Diagram](image)

**Figure 2:** From [Witkin, 1983]

- A first application: look for a local extrema at a given scale and downscale
Link with the heat equation

- The diffusion equation (in 1-D): 1-D:

\[ \frac{\partial}{\partial t} L(x, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} L(x, t) \]  
\[ L(x, 0) = f(x) \text{ (initial condition)} \]  

- This equation rules the heat diffusion (\( L \) is a temperature) in an homogeneous 1-D medium

**Theorem**

A solution of system (3,4) is given by:

\[ L(x, t) = g_{\sqrt{t}} \ast f(x) \]

where \( g \) is the Gaussian function with standard deviation of \( \sqrt{t} \)
Proof

• Easy to verify that $g_{\sqrt{t}}$ is solution of (3):

\[
g(x, t) = \frac{1}{t} e^{-\frac{x^2}{2t}}
\]

\[
\frac{\partial}{\partial t} g(x, t) = -\frac{1}{2t^2} e^{-\frac{x^2}{2t}} + \frac{1}{t} \left( \frac{x^2}{2t^2} \right) e^{-\frac{x^2}{2t}} = \frac{1}{2t^2} e^{-\frac{x^2}{2t}} \left[ \frac{x^2}{t} - 1 \right]
\]

\[
\frac{\partial}{\partial x} g(x, t) = -\frac{x}{t^2} e^{-\frac{x^2}{2t}}
\]

\[
\frac{\partial^2}{\partial x^2} g(x, t) = -\frac{1}{t^2} e^{-\frac{x^2}{2t}} - \frac{x}{t^2} \left( -\frac{x}{t} \right) e^{-\frac{x^2}{2t}} = \frac{1}{t^2} e^{-\frac{x^2}{2t}} \left[ \frac{x^2}{t} - 1 \right]
\]

\[
\Rightarrow \frac{\partial}{\partial t} g(x, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} g(x, t)
\]
• $g_{\sqrt{t}}(x)$ is not solution of eq. (4) (indetermination for $t = 0$)

**Lemma**

\[
\frac{\partial}{\partial x} (f \ast g)(x) = f \ast \frac{\partial g}{\partial x}(x)
\]

\[
f \ast (g + h) = f \ast g + f \ast h
\]

• replace $L$ by $f \ast g_{\sqrt{t}}$ in equation (3), apply the lemma, and remark that $f \ast g$ is solution of eq. (3):

\[
\frac{\partial}{\partial t} (f \ast g_{\sqrt{t}}) - \frac{1}{2} \frac{\partial^2}{\partial x^2} (f \ast g_{\sqrt{t}}) = f \ast \frac{\partial}{\partial t} g_{\sqrt{t}} - \frac{1}{2} f \ast \frac{\partial^2}{\partial x^2} g_{\sqrt{t}}
\]

\[
= f \ast (\frac{\partial}{\partial t} g_{\sqrt{t}} - \frac{1}{2} \frac{\partial^2}{\partial x^2} g_{\sqrt{t}})
\]

\[
= 0
\]
• Is eq. (4) verified?
• \( g_{\sqrt{t}} \ast f(x) = \int_{\mathbb{R}} f(x - u)g_{\sqrt{t}}(u)du \)
• Admitted:
  \[
  \lim_{t \to 0} \int_{\mathbb{R}} f(x) \frac{1}{t\sqrt{2\pi}} e^{-\frac{x^2}{2t}} \, dx = \int_{\mathbb{R}} f(x)\delta(x) \, dx
  \]
  for all function \( f \) integrable in \( \mathbb{R} \)
• Dirac delta function (\( \delta \)) is a generalized function (distribution) implicitly defined by:
  \[
  \int_{\mathbb{R}} f(x)\delta(x) \, dx = f(0)
  \]
• by definition of \( \delta \), it comes:
  \[
  \lim_{t \to 0} \int_{\mathbb{R}} f(x - u)g_{\sqrt{t}}(u)du = \int_{\mathbb{R}} f(x - u)\delta(u) \, du = f(x)
  \]
Proof (lemma)

- Distributivity:
  \[
  \frac{\partial}{\partial x} (f \ast g)(x) = \frac{\partial}{\partial x} \int_\mathbb{R} f(y)g(x - y)dy \\
  = \int_\mathbb{R} \frac{\partial}{\partial x} (f(y)g(x - y))dy \\
  = \int_\mathbb{R} f(y) \frac{\partial}{\partial x} g(x - y)dy
  \]

- Linearity:
  \[
  f \ast (g + h) = \int_\mathbb{R} f(x - y)(g(y) + h(y))dy \\
  = \int_\mathbb{R} f(x - y)g(y)dy + \int_\mathbb{R} f(x - y)h(y)dy
  \]

- Recall that convolution is commutative:
  \[
  f \ast g(x) = \int_\mathbb{R} f(x - y)g(y)dy = \int_\mathbb{R} f(y)g(x - y)dy
  \]
  by change of variable (domain reversal)
Causality principle

- A fundamental principle is verified: non creation of "structures" when the scale increases
- Hypothesis: a structure in an image is characterized by its edges, and then, by its image value extrema
  - at scale $t_0$, if $x_0$ is a local maxima, then
    \[ \frac{\partial^2}{\partial x^2} L(x_0, t_0) < 0 \Rightarrow \frac{\partial}{\partial t} L(x_0, t_0) < 0 \]
  - Taylor expansion at $t$: $L(x_0, t) \sim L(x_0, t_0) + \frac{\partial}{\partial t} L(x_0, t_0)(t - t_0)$
  - then: $\forall t > t_0, \quad L(x_0, t) < L(x_0, t_0)$
  - at scale $t_0$, if $x_0$ local minimum, then
    \[ \nabla^2 L(x_0, t_0) > 0 \Rightarrow \frac{\partial}{\partial t} L(x_0, t_0) > 0 \]
  - then $\forall t > t_0, \quad L(x_0, t) > L(x_0, t_0)$
- Local maxima decrease, local minima increase: the image tends to be more and more homogeneous
Properties of the Gaussian kernel

- It is a smoothing filter: it does not create new image structures
- It is a weighted mean filter:
  \[
  \int_{\mathbb{R}} g_\sigma(x) dx = 1 \tag{5}
  \]
- It has fast decreasing toward zero: beyond \( |x| > \sqrt{t} \), \( g(x) \approx 0 \)
  \( \Rightarrow \) image structures smaller than \( \sqrt{t} \) are suppressed
Gaussian kernel graph
The family of Gaussian functions \((F, g\star)\) has a structure of semi-group, i.e.:

\[ g(., t) \star g(., s) = g(., t + s) \]

A representation at scale \(t_2\) can be deduced from any inferior scale \(t_1\):

\[ t_2 > t_1, L(t_2, x) = g_{t_2 - t_1} \star L(t_1, x) \]

This property is used in the pyramid of resolution.

In frequency domain, we have:

\[ \hat{L}_{t_2}(w) = \hat{g}_{t_2 - t_1}(w) \times \hat{L}_{t_1}(w) \]
Concluding remarks

• Gaussian filter: details smaller than a scale disappear
• Question: does it exist another filters than Gaussian?
• A desirable property: the causality property (no new image structures at higher representation scale)
• In particular: noise should decrease with the scale, and should never be amplified
• Causality is a central property of the scale space theory, see [Koenderink, 1984]
• How to generalize to $\mathbb{R}^2$?
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Continuous linear scale space in $\mathbb{R}^2$

**Definition (Continuous scale space for an image)**

Let $I : \mathbb{R}^2 \to \mathbb{R}$ be an image. Let $L$ be a family of images, derived from $I$, such as:

$$
L : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}
$$

$$
L(x, 0) = I(x), \forall x \in \mathbb{R}^2
$$

$$
L(x, t) = g\sqrt{t} \ast I(x), \forall x \in \mathbb{R}^2, t \in \mathbb{R}^+ \quad (6)
$$

with:

$$
g_{\sqrt{t}}(x) \equiv \frac{1}{2\pi t} e^{-\frac{x^2+y^2}{2t}}
$$

$$
f \ast g(x) \equiv \int_{\mathbb{R}^2} g(x, y) f(x' - x, y' - y) dx' dy'
$$

$L$ is called *scale representation* of the image $I$, parameter $t$ is the scale.
Figure 3: Example of scale decomposition

- Larger structures are suppressed with higher scales...
Figure 4: Animating scale space
The 2-D Gaussian function

- The 2-D Gaussian function is separable:

\[ g_\sigma(x) = g_\sigma(x) \times g_\sigma(y) \]
\[ = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \times \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} = \frac{1}{2\pi t} e^{-(x^2+y^2)/2t} \]

- Implementation of 2-D Gaussian smoothing is efficient:

\[ I \ast g_\sigma(x) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} I(x-x', y-y') g_\sigma(x') dx' \right) g(y') dy' \]

- This property holds for superior dimensions
Theorem

The scale-space representation $L$ of a function $f : \mathbb{R}^2 \to \mathbb{R}$ verifies the following system of equations:

$$\frac{\partial}{\partial t} L(x, t) = \frac{1}{2} \nabla^2 L(x, t)$$

(7)

$$L(x, 0) = I(x)$$

(8)

- $\nabla^2 L = (\nabla^T \nabla) = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} L$ is the Laplacian operator

- System (7,8) is the Heat equation in a 2-D homogeneous medium
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Differentiation in scale space

- Let’s consider an image at scale $t$, and its partial derivative respect to spatial coordinate:

$$
\frac{\partial}{\partial x} L(x, t) = \frac{\partial}{\partial x} (I \ast g_{\sqrt{t}}(x)) = I \ast \frac{\partial}{\partial x} g_{\sqrt{t}}(x)
$$

- Function $g_{\sqrt{t}}$ is $C^\infty$ and $\frac{\partial^n}{\partial x^n} g_{\sqrt{t}}$ is integrable $\forall n$

- If $I$ is integrable, and not necessarily differentiable, then $\frac{\partial^n}{\partial x^n} L$ is defined

- Allows a weak definition of derivative, that can be applied to any functions even not differentiable:

$$
I \ast g_\sigma \xrightarrow{\sigma \to 0} I
$$

$$
I \ast \frac{\partial^n}{\partial x^n} g_\sigma \xrightarrow{\sigma \to 0} \frac{\partial^n}{\partial x^n} I
$$
Properties of derivatives in scale space

- Space scale properties apply to image derivatives
- Indeed: if $L$ solution of $\frac{\partial}{\partial t} L = \frac{1}{2} \nabla^2 L$ then $\partial L$ also a solution
- Moreover:

$$g\sqrt{t_1} \ast \frac{\partial^n}{\partial x^n} g\sqrt{t_2} = \frac{\partial^n}{\partial x^n} g\sqrt{t_1 + t_2}$$  \hspace{1cm} (9)$$

$$\frac{\partial^m}{\partial x^m} g\sqrt{t_1} \ast \frac{\partial^n}{\partial x^n} g\sqrt{t_2} = \frac{\partial^{n+m}}{\partial x^{n+m}} g\sqrt{t_1 + t_2}$$  \hspace{1cm} (10)$$

- If a computer vision problem is formulated as solution of linear PDEs, solutions live in a scale-space
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APPENDIX
• Let us consider an *axiomatic* definition of scale space: the set of parametric functions derived from an image $I$ and an operator $H_t$ such as

1. linearity: $I \mapsto L_t = H_t(I)$ is linear
2. translation invariant system: $H(I \circ T) = H(I) \circ T$, $T$ translation, $\circ$ composition operator
3. semi-group structure: $H_{t_1 + t_2} = H_{t_1} \circ H_{t_2}$

• What operators $H$ may be available?

**Theorem (Uniqueness of Gaussian kernel)**

*All continuous space scale representation writes:*

$$L(x, t) = h_t \ast I(x) = \int_{\Omega} h_t(x') f(x - x') dx'$$  \hspace{1cm} (11)

*with $h_t$ the Gaussian function of variance $t$*
Uniqueness of continuous scale space representation (2)

- Axioms 1. (linearity) and 2. (translation invariant system) leads to a convolution with a kernel to be determined.
- Translation invariant: a change of variable $x \rightarrow x - a$ under the integral (convolution) leaves unchanged the spatial domain ($\mathbb{R}^2$).
- Uniqueness relies on the Pi-theorem (admitted).
Lemma (Pi-theorem (Vaschey-Buckingham))
Consider any physical system $f$ relying $n$ state variables (with dimension) such as $f(q_1, \cdots, q_n) = 0$.
Then it exists a family of variables $\pi_i$ without dimension and a function $F$ such as:

$$
\begin{align*}
F(\pi_1, \cdots, \pi_p) &= 0 \\
\pi_i &= \prod_{k=1}^{n} q_k^{l_k}
\end{align*}
$$

(12)

Variables $\pi_i$ are without dimension as explained as a product (Pi) of variables $q_i$.

- Pi-theorem means that a physical system never depend on the choice of units (and the scale)
- Main lines of the proof are given in appendix
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function g = gD( f, scale, ox, oy)
% Perform a gaussian derivative convolution
% f: input image
% scale: smoothing parameter
% ox, oy: x and y derivate order (0,1,2,3)

% Build the gaussian kernel
K = ceil(3*scale);
x = -K:K;
Gs = exp(-x.^2/(2*scale^2));
Gs = Gs/sum(Gs);

% Calculate the derivatives en x and y—direction
Gsx = gDerivative(ox,x,Gs,scale);
Gsy = gDerivative(oy,x,Gs,scale);

% Do the convolution
g = convSepBrd( f, Gsx, Gsy);
function r = gDerivative(order, x, Gs, scale)

% Compute a derivative of a 1D Gaussian kernel
% order: order derivation 0,1,2,3
% Gs: discretized, centered gaussian kernel
% scale: variance of Gs

switch order
    case 0
        r = Gs;
    case 1
        r = -x/(scale^2).*Gs;
    case 2
        r = (x.^2-scale^2)/(scale^4).*Gs;
    case 3
        r = x.*(3*scale^2-x.^2)/(scale^6).*Gs;
    otherwise
        error('only derivatives up to third order are supported');
end
function g = convSepBrd( f, w1, w2)
%%% convolve along columns and rows with repetition
%%% of the border

N = size(f,1);
M = size(f,2);
K = (size(w1(:),1) − 1)/2;
L = (size(w2(:),1) − 1)/2;
%%% 1 1 1 1 1 | 1 2 ... N | N N N N N
%%% K fois        K fois
iind = min(max((1:(N+2*K)) − K,1),N);
jind = min(max((1:(M+2*L)) − L,1),M);
%%% f(1,..) K fois puis f(1,..) ... f(N,..)
%%% puis f(N,..) K fois
%%% donc repetition des bords ...
fwb = f(iind,jind);
g = conv2(w1,w2,fwb,'valid');
Figure 5: phone=imread("phone.pgm");imshow(phone)
Figure 6: imshow(gD(phone,1,0,0)
Figure 7: imshow(gD(phone,2,0,0))
Examples

Figure 8: imshow(gD(phone,5,0,0)
Examples

Figure 9: imshow(gD(phone,10,0,0))
Figure 10: imshow(gD(phone,1,1,0))
Figure 11: imshow(gD(phone,2,1,0))
Examples

**Figure 12:** imshow(gD(phone, 5, 1, 0))
**Figure 13:** imshow(gD(phone,10,1,0))
Figure 14: imshow(gD(phone,1,0,1))
Figure 15: imshow(gD(phone,2,0,1))
Figure 16: imshow(gD(phone,5,0,1))
Figure 17: imshow(gD(phone,10,0,1))
Figure 18: imshow(gD(phone,1,1,1))
Figure 19: imshow(gD(phone,2,1,1))
Figure 20: imshow(gD(phone,5,1,1))
Figure 21: imshow(gD(phone,10,1,1))
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Automatic scale selection (1)

- Scale space techniques can detect objects whose size is known. Practically, this size is not known.
- In the following, we model a 1-D image structure by the following signal:

\[ f(x) = \sin \omega_0 x \]

\( \omega_0 \), pulsation, represents a structure of period \( \frac{1}{\omega_0} \)
- Scale space representation of this signal:

\[
L(x; t) = f \ast g_{\sqrt{t}}(x)
= e^{-\omega_0^2 t/2} \sin \omega_0 x
\]

- We remark that \( t \mapsto |L(., t)|_\infty = e^{-\omega_0^2 t/2} \) is monotone decreasing
Automatic scale selection (2)

- Derived signal:

\[ L^{(m)}(x, t) = \left. \frac{\partial^m L(x, t)}{\partial x^m} \right|_{L^{(m)}(., t)} = \omega_0^m e^{-\omega_0^2 t/2} \]

- Again \( t \mapsto |L^{(m)}(., t)|_\infty \) is monotone decreasing
  \( \rightarrow \) there is no optimal scale to detect a structure of size \( \frac{1}{\omega_0} \)

- [Lindeberg, 1998]: let us introduce the normalized derivative:

\[ \partial_x^\gamma = t^{\gamma/2} \frac{\partial}{\partial x}, \quad \gamma > 0 \]
Automatic scale selection (3)

- Normalized derivative of $L$ at order $m$:
  \[
  L_{\gamma}^{(m)}(x, t) = \left( t^{\gamma/2} \right)^m \frac{\partial^m L}{\partial x^m}
  \]

- The maximum of this signal:
  \[
  |L_{\gamma}^{(m)}(., t)|_\infty = t^{m\gamma/2} \omega_0^m e^{-\omega_0^2 t/2}
  \]

- $t \mapsto |L_{\gamma}^{(m)}(., t)|$ is no more monotone decreasing along scale parameter, and has a unique maximum at:
  \[
  t_{\text{max}} = \frac{\gamma m}{\omega_0^2}
  \]

  It exists an optimal scale to detect a structure of size $\frac{1}{\omega_0}$, the optimal scale value depends on $\omega_0$

- The maximum depends on $\omega_0$:
  \[
  \max(|L_{\gamma}^{(m)}(., t)|_\infty) = \frac{(\gamma m)^{\gamma m/2}}{e^{\gamma m/2}} \omega_0^{(1-\gamma)m}
  \]  \hspace{1cm} (13)
Automatic scale selection (4)

- Practically, we get $\gamma = 1$: a special case for where the maximum of $L^{(m)}_{\gamma}$ is independent of $\omega_0$

**Figure 22:** Maximal amplitude of $L^{(m)}_{\gamma}$
Part 1: linear scale space

Introduction
Earlier works: image decomposition
Continuous linear scale space: 1-D case
Continuous linear scale space in \( \mathbb{R}^2 \)
Differentiation in scale space
Uniqueness of continuous scale space representation
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Concluding remarks on the linear continuous case

Part 2: non linear scale spaces

APPENDIX
- A discrete image is a constant piecewise function and a set of relatively homogeneous regions.
- Applying a Gaussian smoothing, regions get a Gaussian profile.

**Figure 23:** A (Gaussian) blob
• Laplacian of Gaussian: Laplacian operator localizes the extrema of Gaussian blob both in space and in scale

**Figure 24:** Action of Laplacian on a blob, local extrema in space and scale
Automatic scale selection for blob detection (1)

• Algorithm:
  • Compute $\partial_{xx} L$, $\partial_{yy} L$, $\partial_{xy} L$ for various scales
    Practically, get $\gamma = 1$, then $\partial_{xx} L = tL_{xx}$, $\partial_{yy} L = tL_{yy}$, and $\partial_{xy} L = tL_{xy}$
  • Blob detection: compute
    \[
    F(t) = \text{trace } H(L) = t(L_{xx} + L_{yy})
    \]
    or
    \[
    F(t) = \text{det } H(L) = t(L_{xx}L_{yy} - L_{xy}^2)
    \]
    for all scales
  • Localize local maxima of $F$ both in space and in scale. The optimal scale informs on blob size
Figure 25: Sunflower field
Automatic scale selection for blob detection (3)

Figure 26: Circle diameters are determined from the optimal scale (credit: Hailin Shi)
Edge detection

- First order detectors (approximation of first derivative)
  - Local maxima of gradient norm at various scales
  - Local maxima of normalized gradient norm both in space and scale

- Second order detectors:
  - Marr operator (Laplacian) at various scales and get zeros crossing
  - No automatic scale selection
Harris detector (recall)

- Consider the following tensor applied on an image $I$:

$$A(I)(x, y) = \nabla I^T(x, y) \nabla I(x, y) = \begin{pmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{pmatrix}$$

- Practically, we consider $A(I) = G_{\sigma_I} \ast (\nabla I^T(x, y) \nabla I(x, y))$ with $G_{\sigma_I}$ Gaussian function of variance $\sigma_I^2$ otherwise determinant of $\nabla I^T \nabla I$ is equal to zero

- $\sigma_I = \textit{integration scale}$: determines the “size” of corners to be detected

- Let’s consider $\kappa \in [0.04, 0.15]$, Harris operator is defined by:

$$R(I) = \det(A(I)) - \kappa \text{Tr}^2(A(I))$$

- $(x, y)$ is a corner $\iff R(I)(x, y) > 0$ and $(x, y)$ is a local maxima of $R$
Harris-Laplace detector

- Harris detector is rotation invariant as $R(I) = \lambda_1 \lambda_2 - \kappa (\lambda_1 + \lambda_2)^2$, $\lambda_1$ and $\lambda_2$ eigenvalues of $A(I)$
- $\sigma_I$ is not a scale parameter: $R$ is not scale invariant if a mono-scale approximation of image derivative (Sobel for instance) is used
- Harris-Laplace replaces $I(x, y)$ by $L(x, y, \sigma_D) = G_s \ast I(x, y)$ in the definition of $A$
- Normalized derivatives may also be used ($L_x(x, y, \sigma_D)$, $L_x(x, y, \sigma_D)$)
- $\sigma_D = \text{differentiation scale}$
- Harris-Laplace detector: $(x, y, \sigma_D)$ is a corner
  $\Leftrightarrow R(I)(x, y, \sigma_I, \sigma_D) > 0$ and $(x, y)$ is a local maximum of $R$ both in space and differentiation scale for a given integration scale $\sigma_I$
Harris-Laplace detector and automatic selection of the integration scale

- How to choice $\sigma_I$?
- Algorithm [Lindeberg, 1998]:
  1. $k = 1$, $s_1$ set to an initial scale integration value
  2. $(x_i, y_i)$ the set of detected corners by Harris-Laplace at $\sigma_I = s_1$
  3. $k = k + 1$
  4. Find the scale maximizing the normalized Laplacian of Gaussian of points $(x_i, y_i)$

  $$\hat{t} = \arg\max_{t \in [0.7, \ldots, 1.4]} |ts_k(L_{xx}(x_i, y_i, ts_k) + L_{yy}(x_i, y_i, ts_k))|$$

  5. $s_{k+1} = \hat{t}s_k$, determine $(x_i, y_i)$ the set of corners detected by Harris-Laplace at scale integration $s_{k+1}$
  6. Iterate 3-4 up to convergence of $s_k$
 Others multi-scale detectors

- Scale invariant features detectors: SIFT, SURF, ... direct application of space-scale theory
- Not covered here but already studied in other M2 courses
Part 1: linear scale space

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Part 2: non linear scale spaces

APPENDIX
• Solutions of the homogeneous heat equation can be expressed as a Gaussian smoothing
• Gaussian smoothing allows to select the primitives of a certain scale (i.e. size). This scale depends on the standard deviation of the Gaussian kernel
• Scale selection can be done optimally (using normalized derivatives)
• Scale space can be defined in an axiomatic way (linearity, shift invariant, semi-group) and is unique (Gaussian smoothing)
Perspectives: what’s happen in the discrete case?

- Discrete in space, continuous in scale?
  - Important question: practically, the Gaussian kernel is sampled for a given scale.
  - A scale space representation is called *admissible* if:
    - is linear and shift invariant (i.e. discrete convolution)
    - *respects the causality principle* (i.e. low pass filter)
    - verify the semi-group property
  - The set of admissible kernels is a countable family and has been fully described by Schoenberg (1953)
  - A direct consequence of Schoenberg: the Gaussian kernel sampling is not an admissible representation: the semi-group property is not respected excepted when $t_1$ divides $t_2$
Another consequence: the closest infinite impulse response (IIR) admissible filter to sampled Gaussian filter is the \textit{discrete analog of Gaussian kernel}

\[ T(n, t) = e^{-\alpha t} I_n(\alpha t), \alpha > 0 \]

with \( I_n(t) = (-i)^n J_n(it) \) and \( J_n \) Bessel function of first kind

Part 2: non linear scale spaces
Part 1: linear scale space

Part 2: non linear scale spaces

Motivations

Digression: discretization of PDEs
Perona & Malik diffusion
Anisotropic diffusion
Edge Enhancing
Coherence Enhancing
Image restoration
To go further

APPENDIX
Motivations (1)

- Linear scale space representation has desirable properties but also has some issues
  - loss of important details (contours) at higher scales
  - loss of localization and lower accuracy
- For some applications it’s a concern!
  - denoising
  - segmentation
  - regularization
- Principle of non linear scale spaces: find representations that respect the causality principle AND that preserve image discontinuities (two antagonist constraints)
Motivations (2)

- Basic idea: consider a diffusion process **image driven**:
  - Over edges: no diffusion (contours are preserved)
  - Elsewhere: diffusion (smoothing)
- Equivalence between linear scale space and homogeneous heat equation (linear diffusion PDE)
- What’s about non linear diffusion PDEs? Is it possible to derive alternative scale spaces?
- Roadmap: consider a non linear diffusion PDE and check if some scale spaces axioms are verified (causality property, edges preservation)
Part 1: linear scale space

Part 2: non linear scale spaces

Motivations

Digression: discretization of PDEs
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APPENDIX
Discretization of PDEs

• Numerous methods and depending on type of equation
• Here we present the finite difference method that applies on
  • elliptic equations (such as Poisson equation: $u_{xx} + u_{yy} = f$)
  • hyperbolic equations (such as waves equation: $u_{tt} = cu_{xx}$)
  • and finally parabolic equations (such as diffusion: $u_t = (D_x u)_x$)
• In the following we denote $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, \ldots
• Recall: Taylor expansion of a function $f$ at point $x_0$:

$$f(x) \approx \sum_{i=0}^{n} f^{(i)}(x_0) \frac{(x - x_0)^i}{i!} + O((x - x_0)^n)$$

with $f^{(i)}$ the $i$-order derivative, $O(u) \xrightarrow{u \to 0} 0$
• Approximation at order $i$:

$$f(x) \approx \sum_{i=0}^{n} f^{(i)}(x_0) \frac{(x - x_0)^i}{i!}$$

stands only in a neighborhood of $x_0$
Approximation of first derivative

- **Example 1 (first order Taylor expansion):** forward difference

\[
 f(x + \triangle) \approx f(x) + f'(x)\triangle \\
 f'(x) \approx \frac{f(x + \triangle) - f(x)}{\triangle}
\]

(14)

- **Example 2 (first order Taylor expansion):** backward difference

\[
 f(x - \triangle) \approx f(x) - f'(x)\triangle \\
 f'(x) \approx \frac{f(x) - f(x - \triangle)}{\triangle}
\]

(15)

- **Example 3:** centered difference (subtracting (15) to (14))

\[
 f(x + \triangle) - f(x - \triangle) \approx 2f'(x)\triangle \\
 f'(x) \approx \frac{f(x + \triangle) - f(x - \triangle)}{2\triangle}
\]

(16)

(17)
Approximation of second derivative

- Consider the two following Taylor expansions at second order:

\[ f(x + \Delta) \approx f(x) + f'(x)\Delta + f''(x)\frac{\Delta^2}{2} \]
\[ f(x - \Delta) \approx f(x) - f'(x)\Delta + f''(x)\frac{\Delta^2}{2} \]

- Adding them leads to

\[ f''(x) \approx \frac{f(x + \Delta) - 2f(x) + f(x - \Delta)}{\Delta^2} \]

- Others schemes are possible: remark that \( f''(x) = (f'(x))' \) and approximate in two steps using first order operators previously seen

- Important remark: loss of uniqueness in the discrete world
The question: how to approximate the heat equation in a \textit{correct} numerical scheme?

\[
\frac{\partial u}{\partial t}(x, t) = c \frac{\partial^2 u}{\partial x^2}(x, t) \quad x \in [x_0, x_1], \quad t \in [t_0, t_1] \quad (18)
\]

Space discretization: \( x \mapsto u(x, t) \) is sampled on an uniform grid

- \( x_j = x_0 + j \times \Delta x \) with \( j = 0 \cdots J \) and \( J = \frac{x_1 - x_0}{\Delta x} \), \( \Delta x \) is called space step

- \( \frac{\partial^2 u}{\partial x^2}(x_j, t) \simeq \frac{u(x_{j+1}, t) - 2u(x_j, t) + u(x_{j-1}, t)}{\Delta_x^2} \)

Time discretization: \( t \mapsto u(x, t) \) is sampled on an uniform grid

- \( t_n = t_0 + n \times \Delta t \), \( \Delta t \) is called time step

- \( \frac{\partial u}{\partial t}(x, t_n) \simeq \frac{u(x, t_{n+1}) - u(x, t_n)}{\Delta t} \)
Application: approximation of the heat equation (1-D)

• Replace left and right members in (18) by their approximation:

\[
\frac{u_j^{n+1} - u_j^n}{\triangle t} = c \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\triangle x^2}
\]

• Numerical scheme called FTCS (Forward Time, Centered Space):

\[
u_j^{n+1} = u_j^n + c \frac{\triangle t}{\triangle x^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n)
\]

(19)

• Given an initial condition (a vector \( U^0 = (u_0^0 \quad u_1^0 \quad \cdots \quad u_J^0)^T \)), we compute an approximate solution of \( u(x^n, t_j) \)

• The scheme is said explicit as we can compute in one pass \( U^n \) from \( U^{n-1} \)
Stability analysis of a numerical scheme

• Does a numerical scheme converge to a correct solution? A necessary condition is *stability*: the error

\[ \epsilon_j^n = u(x_j, t_n) - u_j^n \]

remains small for all \( n \), \( u_j^n \) is a numerical solution of the discrete equation computed with a finite precision, \( u(x_j, t_n) \) is the truth solution of the continuous equation

• A tool: the spectral Fourier analysis

• Consider solutions in the form of \( u(x_j, t_n) = \epsilon_j^n + u_j^n \), replace in (19):

\[ \epsilon_{j}^{n+1} = \epsilon_j^n - \alpha(\epsilon_{j+1}^n - 2\epsilon_j^n + \epsilon_{j-1}^n) \quad (20) \]

with \( \alpha = \frac{c \Delta t}{\Delta x^2} \)

• Error verifies the discrete PDE
Spectral Fourier analysis

- Spectral analysis aims to study the behavior of solution writing:

\[ \epsilon_j = \sum_{k=1}^{N/2} \xi_k e^{ikj \Delta x}, \quad j = 0 \ldots J \]

- Error also depends on time but the Fourier basis don’t, \( \xi_k = \xi_k(n) \)
- The numerical scheme, eq. (20), is similar to a geometric sequence, the error evolves exponentially in time:

\[ \epsilon^n_j = \sum_{k=1}^{N/2} \xi^n_k e^{ikj \Delta x}, \quad \text{with } \xi_k \text{ constant} \]

- A sufficient condition is to study the stability for each mode \( k \):

\[ \xi^n_k e^{ikj \Delta x} \]
To summarize: the spectral Fourier analysis of a discrete PDE studies the behavior of solutions writing $u_j^n = \xi_k^n e^{ikj \Delta x}$ when $n$ tends to $\infty$.

Application to the discrete heat equation (FTCS scheme): let’s replace $u_j^n$ by $\xi_k^n e^{ikj \Delta x}$ in Eq. (20):

\[
\begin{align*}
\xi_k^{n+1} e^{ikj \Delta x} &= \xi_k^n e^{ikj \Delta x} + \alpha (\xi_k^n e^{ik(j+1) \Delta x} - 2\xi_k^n e^{ikj \Delta x} \\
&\quad + \xi_k^n e^{ik(j-1) \Delta x}) \\
\xi_k &= 1 + \alpha (e^{i \Delta x k} - 2 + e^{-i \Delta x k}) \\
&= 1 + \alpha (2 \cos(\Delta x k) - 2) \\
&= 1 - 4\alpha \sin^2 \frac{\Delta x k}{2} \quad \text{(recall: $\cos 2a = 1 - 2 \sin^2 a$)}
\end{align*}
\]
Stability of discrete Heat equation (FTCS)

- The scheme is stable iff $|\xi_k| < 1$

- If $\xi_k > 0$:
  \[
  \xi_k = 1 - 4\alpha \sin^2 \frac{\Delta x k}{2} < 1 \quad \forall \alpha > 0
  \]

- If $\xi_k < 0$:
  \[
  \xi_k = 4\alpha \sin^2 \frac{\Delta x k}{2} - 1 < 1 \iff 4\alpha \sin^2 \frac{\Delta x k}{2} < 2
  \iff \alpha < \frac{1}{2}
  \]

- CFL condition\(^1\): the scheme is stable if $\frac{c \Delta t}{\Delta x^2} < \frac{1}{2}$

- Example: with $\Delta x = \Delta t = 1$, the condition is $c < \frac{1}{2}$

- Practically $\Delta x$ is fixed, $\Delta t$ must be chosen in order to respect the CFL condition

\(^1\)Courant-Freidrichs-Lewy
Figure 27: $\alpha = 0.4$, the scheme is stable
Figure 28: $\alpha = 0.5$, the scheme is locally unstable
Stability of discrete Heat equation (FTCS)

Figure 29: $\alpha = 0.6$, the scheme is unstable
Consider now a backward in time and centered in space scheme (BTCS):

\[
\frac{u_j^n - u_j^{n-1}}{\Delta t} = c \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta_x^2}
\]

Leads to the following scheme:

\[
-\alpha u_{j+1}^n + (1 + 2\alpha) u_j^n - \alpha u_{j-1}^n = u_{j-1}^{n-1}
\]  

It is an implicit scheme that requires a matrix inversion: \(AU^n = U^{n-1}\) with \(A\) tri-diagonal

\[
A = \begin{pmatrix}
1 + 2\alpha & -\alpha & 0 & \cdots \\
-\alpha & 1 + 2\alpha & -\alpha & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\hdots & \hdots & \hdots & \hdots & \hdots & \hdots \\
\end{pmatrix}
\]
Discrete Heat equation: implicit scheme

- Error:
  
  $$-\alpha \epsilon_{j+1}^n + (1 + 2\alpha)\epsilon_j^n - \alpha \epsilon_{j-1}^n = \epsilon_{j-1}^{n-1}$$

- Spectral Fourier analysis:
  
  $$\xi_k (-\alpha e^{i\Delta k} + (1 + 2\alpha) - \alpha e^{-i\Delta k}) = 1$$
  
  $$\xi_k = \frac{1}{1 + 2\alpha - 2\alpha \cos(\Delta k)}$$
  
  $$= \frac{1}{1 + 4\alpha \sin^2 \frac{k\Delta}{2}}$$
  
  $$|\xi_k| < 1$$

- The scheme is unconditionally stable
Figure 30: $\alpha = 0.6$, the scheme remains stable
Conclusion on discrete Heat equation

- **Pro:** the implicit scheme (BTCS) is always stable!
- **Cons:**
  - require the inversion of a matrix (possibly huge for 2-D diffusion)
  - practically the inversion of huge matrix relies on iterative algorithms (that do not provide an exact solution)
  - For non linear diffusion, the matrix depends on image configuration and changes at each iteration
- **Conclusion:**
  - if the CFL condition is verified, prefer explicit scheme
  - in other cases, use implicit scheme
Part 1: linear scale space

Part 2: non linear scale spaces

Motivations

Digression: discretization of PDEs

Perona & Malik diffusion

Anisotropic diffusion

Edge Enhancing

Coherence Enhancing

Image restoration

To go further

APPENDIX
Founding work on non linear scale space come from [Perona and Malik, 1990]: “Space scale and Edge Detection Using Anisotropic Diffusion”

The problem is formulated in an axiomatic way:
- causality principle
- edges are localized for all scales
- homogeneous regions are smoothed

This leads to diffusion whose characteristics depend on local configuration of image values

Consequently: the PDE describing such a diffusion is no more linear
Perona & Malik diffusion: formulation

- Consider the following PDE:

\[
\frac{\partial L}{\partial t}(x, t) = \nabla \cdot (c(x, t) \nabla L(x, t)) \quad x \in \mathbb{R}^2, \ t > 0 \quad (22)
\]

\[
= \frac{\partial}{\partial x} \left( c(x, t) \frac{\partial L(x, t)}{\partial x} \right) + \frac{\partial}{\partial y} \left( c(x, t) \frac{\partial L(x, t)}{\partial y} \right)
\]

\[
L(x, 0) = l(x) \quad (23)
\]

- The right member of (22) may write:

\[
\frac{\partial L}{\partial t}(x, t) = c(x, t) \nabla^2 L(x, t) + \nabla c(x, t) \cdot \nabla L(x, t)
\]

- If \( c(x, t) = c \) then \( \nabla c(x, t) = 0 \), and we retrieve the homogeneous heat equation

- \( c \): diffusion coefficient, ruling the "speed" diffusion
  - \( c \approx 0 \): no diffusion (stationary), image is locally preserved
  - \( c \) is constant, homogeneous diffusion, image is locally smoothed
Choice for \( c \) (1)

- According to Perona & Malik: choose \( c \) such as
  - \( c(x, t) = 1 \) over homogeneous regions: strong smoothing
  - \( c(x, t) = 0 \) over edges area: no smoothing
- Let \( E(x, t) \) be an oriented edge map at scale \( t \):
  \[
  E(x, t) = \begin{cases} 
  K\vec{e}'(x, t) & \text{if } x \text{ edge} \\
  0 & \text{otherwise}
  \end{cases}
  \]
- \( \vec{e}' \) vector orthogonal to the edge
- \( K \) a contrast parameter representing the gray scale value difference between regions adjacent to the edge
Choice for \( c \) (2)

- Typically choice: \( c(x, t) = g(\|E(x, t)\|) \), i.e. an *isotropic* diffusion (doesn’t depend on the gradient orientation)
- \( E \) depends on \( L \): equation (22) no more linear
- \( g \) is a fast decreasing function, for example:

\[
1/(1+(x)^4)
\]

- Perona & Malik’s choice for the edges map: \( E(x, t) = \nabla L(x, t) \)
Choice for $g$ (1)

- The solutions of (22) verify the causality principle (derive from a general theorem for parabolic PDEs, see P&M paper)
- Edges enhancement with increasing scale: in 1-D, Equation (22) reduces to

$$\frac{\partial L}{\partial t} = \frac{\partial}{\partial x} \left( c \frac{\partial L}{\partial x} \right)$$

- Behavior of edges: $\frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} L \right) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} L \right) = \frac{\partial^2}{\partial x^2} \left( c \frac{\partial}{\partial x} L \right)$
  - we chose $c = g(L_x)$
  - let's denote $\phi(L_x) = g(L_x)L_x$ (with $L_x = \frac{\partial}{\partial x} I$), $\phi$ is called “flux”

$$\frac{\partial}{\partial t} L_x = \frac{\partial^2}{\partial x^2} \phi(L_x)$$
$$= \frac{\partial}{\partial x} \left( L_{xx} \phi'(L_x) \right)$$
$$= L_{xxx} \phi'(L_x) + L_{xx}^2 \phi''(L_x)$$
Choice for $g$ (2)

- Behavior of edges:
  \[
  \frac{\partial}{\partial t} L_x = L_{xxx} \phi'(L_x) + L_{xx}^2 \phi''(L_x)
  \]

- Let an edge such as $L_x > 0$, then $L_{xx} = 0$, and $L_{xxx} < 0$, and
  \[
  \frac{\partial}{\partial t} L_x = L_{xxx} \phi'(L_x)
  \]

- Two cases:
  1. $\phi'(L_x) > 0 \Rightarrow \frac{\partial}{\partial t} L_x < 0$: $t \mapsto L_x$ is a decreasing function: attenuation of the edges with increasing scale
  2. $\phi'(L_x) < 0 \Rightarrow \frac{\partial}{\partial t} L_x > 0$: enhancement of the edges with increasing scale

- If $\phi$ increases: edges are enhanced and better localized
- But: the slope should be moderated in order to respect the Causality Principle
Choice for $g$ (3)

- The choice of $g$ is guided by the behavior of $\phi$ ($\phi(x) = xg(x)$) to obtain an enhancement of edges.
- An admissible choice for $\phi$, a decreasing function beyond a contrast threshold $K$:

![Graph showing function $x/(1+(x)^4)$ over the range $0$ to $5$.]

**Figure 31**: Choice for $\phi$, $K$ is the absciss of maximal value

- if $|L_x| > K \Rightarrow \frac{\partial}{\partial t} L_x > 0$: edges are enhanced
- if $|L_x| < K$: edges are attenuated
Choice for \( g \) (4)

- Tukey conductivity:
  \[ g(x) = e^{-\left(\frac{x}{K}\right)^2} \]

- Lorentz conductivity:
  \[ g(x) = \frac{1}{1 + \left(\frac{x}{K}\right)^{1+\alpha}} \quad \alpha > 0 \]

- Function close to 1 in the neighborhood of 0 and fast decreasing. Inflection point at \( x = K \).

- Then \( \phi(x) = xg(x) \approx x \) in the neighborhood of 0 and decreasing beyond \( K \).

- Many other choices are possible.
Discretizing P&M equation (1)

- We denote $L(x_0 + i, y_0 + j, k\triangle t) = L_{i,j}^k$, same for $g$ (step space set to 1: pixel unit)
- $\frac{\partial}{\partial t} L \approx \frac{L_{i,j}^{k+1} - L_{i,j}^k}{\triangle t}$
- $\nabla.(g \nabla L) = \nabla g \cdot \nabla L + g \nabla^2 L$
- $\nabla g$ and $\nabla L$ are approximated by a forward difference
- $\nabla^2 L = L_{xx} + L_{yy}$ is approximated by a centered difference
- $g(x, y, t) = g(\|\nabla L(x, y, t)\|): g$ depends on the current time iteration $k$
Discretizing P&M equation (2)

- The right member writes:

\[ g_{i,j}^k (L_{i,j+1}^k + L_{i,j-1}^k + L_{i+1,j}^k + L_{i-1,j}^k - 4L_{i,j}^k) \]

\[ + (g_{i+1,j}^k - g_{i,j}^k)(L_{i+1,j}^k - L_{i,j}^k) + (g_{i,j+1}^k - g_{i,j}^k)(L_{i,j+1}^k - L_{i,j}^k) \]

\[ = g_{i,j}^k (L_{i-1,j}^k - L_{i,j}^k) + g_{i,j}^k (L_{i,j-1}^k - L_{i,j}^k) \]

\[ + g_{i+1,j}^k (L_{i+1,j}^k - L_{i,j}^k) + g_{i,j+1}^k (L_{i,j+1}^k - L_{i,j}^k) \]

- Final numerical scheme (using P&M notation):

\[ L_{i,j}^{k+1} = L_{i,j}^k + \Delta t [C_N \cdot \nabla N L + C_S \cdot \nabla S L + C_E \cdot \nabla E L + C_W \cdot \nabla W L]^k_{i,j} \]

with:

\[ [\nabla N L]^k_{i,j} = L_{i-1,j}^k - L_{i,j}^k \]

\[ [\nabla S L]^k_{i,j} = L_{i+1,j}^k - L_{i,j}^k \]

\[ [C_N]^k_{i,j} = g_{i,j}^k \]

\[ [C_S]^k_{i,j} = g_{i+1,j}^k \]

\[ [\nabla E L]^k_{i,j} = L_{i,j+1}^k - L_{i,j}^k \]

\[ [\nabla W L]^k_{i,j} = L_{i,j-1}^k - L_{i,j}^k \]

\[ [C_E]^k_{i,j} = g_{i,j+1}^k \]

\[ [C_W]^k_{i,j} = g_{i,j}^k \]
Discretizing P&M equation (3): dual grid and simplification

- The function $g = g(\nabla I)$ should also be discretized. Contours can be localized on the dual grid:

  \[
  C_N = g(\|\nabla L^k_{i+\frac{1}{2},j}\|) \quad C_S = g(\|\nabla L^k_{i-\frac{1}{2},j}\|) \\
  C_E = g(\|\nabla L^k_{i,j+\frac{1}{2}}\|) \quad C_W = g(\|\nabla L^k_{i,j-\frac{1}{2}}\|)
  \]

$L^k_{i+\frac{1}{2},j}$ may be obtained by linear interpolation

- P&M simplify as:

  \[
  C_N = g(\|\nabla N u^k_{i,j}\|) \quad C_E = g(\|\nabla E u^k_{i,j}\|) \\
  C_S = g(\|\nabla S u^k_{i,j}\|) \quad C_W = g(\|\nabla W u^k_{i,j}\|)
  \]
Figure 32: Original image
Experimenting Perona & Malik diffusion

Figure 32: Lorentz conductivity, 10 iterations, $K = 20$
Figure 32: Lorentz conductivity, 30 iterations, $K = 20$
Experimenting Perona & Malik diffusion

Figure 32: Lorentz conductivity, 100 iterations, $K = 20$
Importance of parameter $K$

Figure 33: Lorentz conductivity, 100 iterations, $K = 5$
Importance of parameter $K$

**Figure 33:** Lorentz conductivity, 100 iterations, $K = 30$
Instability of the scheme

Figure 34: Lorentz conductivity, 100 iterations, $K = 20$, $\triangle t = 0.5$
Figure 34: Lorentz conductivity, 10 iterations, $K = 20$, $\Delta t = 4$
Conclusion on P&M diffusion

- Isotropic diffusion guided by image configurations: image is preserved over edges, elsewhere image is smoothed.
- The scheme is explicit and then unstable for some choices of parameters.
- A semi-implicit and regularized scheme may be found in [Catté et al., 1992].
Part 1: linear scale space

Part 2: non linear scale spaces

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Anisotropic diffusion
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To go further

APPENDIX
Anisotropic diffusion

- **Isotropic diffusion**: diffusion does not depend on the direction (but direction of what?)
- **Anisotropic**: diffusion depends on the direction
- **Physical principles**:
  - the same equations describe the diffusion of heat and the diffusion of chemical species
  - particles in a high concentration areas migrate to the lower concentration area: diffusion has a direction
Anisotropic diffusion: again inspired by physics

- Fick’s laws:
  1. Flux is the direction of matter transport
  2. Variation of heat or concentration (of a given chemical specie) is equal to the balance of incoming and outgoing fluxes

- Mathematical formalization for $u : \mathbb{R}^3 \rightarrow \mathbb{R}$:

$$\frac{\partial u}{\partial t} = -\nabla \cdot (D \nabla u)$$  \hspace{1cm} (24)

- $D$ is a tensor of diffusion (a symmetric and positive definite matrix of size $3 \times 3$)
2-D anisotropic diffusion

- For an image: \( L : \mathbb{R}^2 \rightarrow \mathbb{R} \), let's consider:

\[
\frac{\partial L}{\partial t} = \nabla.(D \nabla L)
\]  

(25)

- If \( D = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \) \( \Rightarrow \) isotropic diffusion:

\[
\frac{\partial L}{\partial t} = \nabla.(cI_d \nabla L) = \nabla.(c \nabla L)
\]

- \( D \) is symmetric positive definite, meaning that:

\[
D = R \Lambda R^T \text{ with } \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ and } \lambda_{1,2} \geq 0
\]

  - \( R \) is an orthogonal matrix representing a change of basis: from the canonical basis to the basis of eigenvectors \((\vec{v}_1, \vec{v}_2)\) of \( D \)
  - \( R \) is a rotation mapping the canonical vector \( \vec{i} \) to \( \vec{v}_1 \), and \( \vec{j} \) to \( \vec{v}_2 \)
  - \( \lambda_{1,2} \) are the associated eigenvalues
The tensor $D$

- $D$ is a linear mapping; applied on $\vec{v}_1$, it comes:

$$
D\vec{v}_1 = R \Lambda R^T \vec{v}_1 = R \Lambda \vec{i} \\
= R \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda_1 R \vec{i} \\
= \lambda_1 \vec{v}_1
$$

remember that $R^T = R^{-1}$

- and: $D\vec{v}_2 = \lambda_2 \vec{v}_2$
  - vectors in the direction of $\vec{v}_1$ are scaled by $\lambda_1$
  - vectors in the direction of $\vec{v}_2$ are scaled by $\lambda_2$
  - Any vector $\vec{u}$ writes $\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2$ and then $D\vec{u} = \lambda_1 c_1 \vec{v}_1 + \lambda_2 c_2 \vec{v}_2$
How to choice $D$? A first try

- Idea: in the basis $(\vec{v}_1, \vec{v}_2)$ one can control the diffusion in the direction of image gradient and edges

- Get $R$ as:

$$
R = \frac{1}{\|\nabla L\|} \begin{pmatrix} L_x & -L_y \\ L_y & L_x \end{pmatrix}
$$

i.e. a rotation of angle the direction of $\nabla L$

- but in Eq. (25) $D$ applies on $\nabla L$:

$$
D \nabla L = R \Lambda R^T \nabla L = R \Lambda \vec{i} = \lambda_1 \nabla L
$$

and then $\frac{\partial L}{\partial t} = \lambda_1 \nabla^2 L$

- This is the linear isotropic diffusion! Bad idea!
Part 1: linear scale space

Part 2: non linear scale spaces

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APPENDIX
Edge Enhancing [Weickert, 1998]

- Consider $\nabla_\sigma L$ (gradient of $u$ at scale $\sigma > 0$), the “local” gradient:

$$D_\sigma = R_\sigma \Lambda R_\sigma^T$$

$$R_\sigma = \frac{1}{||\nabla_\sigma L||} \begin{pmatrix} L_\sigma^x & -L_\sigma^y \\ L_\sigma^y & L_\sigma^x \end{pmatrix}$$

with $L_\sigma^x = L \ast \frac{\partial G_\sigma}{\partial x}$

- Let’s apply $D_\sigma$ on $\nabla L$:

$$D_\sigma \nabla L = \frac{1}{||\nabla_\sigma L||^2} \left( \lambda_1 (u_\sigma^x)^2 + \lambda_2 (u_\sigma^y)^2 (\lambda_1 - \lambda_2) L_\sigma^x L_\sigma^y \right)$$

$$D_\sigma \nabla L = \frac{1}{||\nabla_\sigma L||^2} \left( \lambda_1 \nabla_\sigma L \nabla_\sigma L^T + \lambda_2 \nabla_\sigma^\perp L \nabla_\sigma^\perp L^T \right) \nabla L$$

with $\nabla_\sigma^\perp L = \begin{pmatrix} L_y \\ -L_x \end{pmatrix}$

- In the general case: $\nabla_\sigma^\perp L^T \nabla L \neq 0$
Edge Enhancing (2)

- Eigenvectors of $D_\sigma$:
  \[ \vec{v}_1 = \frac{\nabla_\sigma u}{\|\nabla_\sigma u\|} \text{ and } \vec{v}_2 = \vec{v}_1^\perp \]

- Expression of $\nabla L$ in the basis of eigenvalues of $D$:
  \[ \nabla L = c_1 \vec{v}_1 + c_2 \vec{v}_2 \]

- with $\vec{v}_1 \cdot \nabla L = c_1$ and $\vec{v}_2 \cdot \nabla L = c_2$

- then: $R_\sigma^T \nabla L = \begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{pmatrix} \nabla L = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

- and: $\Lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \lambda_1 \\ c_2 \lambda_2 \end{pmatrix}$ and $R_\sigma^T \begin{pmatrix} c_1 \lambda_1 \\ c_2 \lambda_2 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} c_1 \lambda_1 \\ c_2 \lambda_2 \end{pmatrix}$

- finally:
  \[ D_\sigma \nabla L = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 \]
• \((c_1, c_2)\) characterizes \(\nabla L\)
• \((\lambda_1, \lambda_2)\) characterizes \(D_\sigma\):
  1. if \(\sigma\) close to 0: \(\nabla_\sigma L \rightarrow \nabla L\) and \(c_1 \rightarrow 1\) and \(c_2 \rightarrow 0\): the diffusion is almost isotropic, diffusivity is ruled by \(\lambda_1\)
  2. if \(\sigma\) is high, in general \(\nabla_\sigma L\) is not colinear with \(\nabla L\) and \(c_2 \gg 0\): the diffusion direction depends on values of \(\lambda_1\) and \(\lambda_2\)
  3. if \(\lambda_1 = \lambda_2\), then \(D\nabla L = \lambda_1(c_1 \vec{v}_1 + c_2 \vec{v}_2) = \lambda_1 \nabla L\): the diffusion is again isotropic!
  4. if \(\lambda_1 = 0\) and \(\lambda_2 \gg 0\), then \(\nabla_\sigma L \perp \nabla L\): the diffusion direction is oriented along edges, diffusion is not more isotropic
Edge Enhancing (4)

- to summarize:
  - $\lambda_1$ tunes diffusion in the direction of $\nabla_\sigma L$
  - $\lambda_2$ tunes diffusion in the orthogonal direction of $\nabla_\sigma L$
  - if $\lambda_1 \geq \lambda_2$, diffusion is closely isotropic
  - if $\lambda_1 < \lambda_2$, diffusion is anisotropic

- Possible choice for $\lambda_1$ et $\lambda_2$ ([Weickert, 1998]):
  - $\lambda_2 = e^{-\frac{\|\nabla_\sigma L\|^2}{k^2}}$
  - $\lambda_1 = \frac{1}{5}\lambda_2 \ll \lambda_2$
    - $\Rightarrow$ diffusion is oriented along edges
  - the diffusion is non linear w.r.t. to $L$ (why?)
Edge Enhancing (5)

- Image is smoothed along edges (high value of $\lambda_2$) ...
- ... and not in the direction of $\nabla_\sigma L$ ($\lambda_1 \approx 0$)
- In homogeneous regions: $\lambda_1 \approx 0$, $\lambda_2 \approx 0$: no diffusion, image is preserved
Discretizing Edge Enhancing (1)

- Let \( D = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \)

\[
L_t = \nabla \cdot (D \nabla L) = \frac{\partial}{\partial x} \left( a \frac{\partial L}{\partial x} \right) + \frac{\partial}{\partial x} \left( b \frac{\partial L}{\partial y} \right) + \frac{\partial}{\partial y} \left( b \frac{\partial L}{\partial x} \right) + \frac{\partial}{\partial y} \left( c \frac{\partial L}{\partial y} \right)
\]

- Terms \( \frac{\partial}{\partial x} (a \frac{\partial u}{\partial x}) \) et \( \frac{\partial}{\partial y} \left( c \frac{\partial u}{\partial y} \right) \) had already been addressed by P&M

- Remains the crossed terms:
  - \( b \frac{\partial u}{\partial y} \approx b_{i,j} \frac{L_{i,j+1} - L_{i,j-1}}{2} = f_{i,j} \)
  - \( \frac{\partial}{\partial x} \left( b \frac{\partial L}{\partial y} \right) \approx \frac{f_{i+1,j} - f_{i-1,j}}{2} = \)
    \[
    \frac{1}{4} \left( b_{i+1,j} (L_{i+1,j+1} - L_{i+1,j-1}) - b_{i-1,j} (L_{i-1,j} - L_{i-1,j-1}) \right)
    \]
  - same discretization of the second crossed term

- These choices lead to a stable numerical scheme
After factoring according to $L$ terms, one finds:

$$L_{i,j}^{k+1} = u_{i,j}^k + \Delta t \left[ -\frac{b_{i-1,j} + b_{i,j+1}}{4} L_{i-1,j+1}^k + \frac{c_{i,j+1} + c_{i,j}}{2} L_{i,j+1}^k 
+ \frac{b_{i+1,j} + b_{i,j+1}}{4} L_{i+1,j+1}^k + \frac{a_{i-1,j} + a_{i,j}}{2} L_{i-1,j}^k 
- \frac{a_{i-1,j} + 2a_{i,j} + a_{i+1,j} + c_{i,j-1} + 2c_{i,j} + c_{i,j+1}}{2} L_{i,j}^k 
+ \frac{a_{i+1,j} + a_{i,j}}{2} L_{i+1,j}^k + \frac{b_{i-1,j} + b_{i,j-1}}{4} L_{i-1,j}^k 
+ \frac{c_{i,j-1} + c_{i,j}}{2} L_{i,j-1}^k - \frac{b_{i+1,j} + b_{i,j-1}}{4} L_{i+1,j-1}^k \right]$$
Plugging $D_\sigma$ into the numerical scheme

- Recall:

$$D_\sigma = R_\sigma \Lambda R_\sigma^T$$

$$D_\sigma = \frac{1}{\|\nabla_\sigma L\|^2} \begin{pmatrix}
\lambda_1(L_x^{\sigma})^2 + \lambda_2(L_y^{\sigma})^2 & (\lambda_1 - \lambda_2)L_x^{\sigma}L_y^{\sigma} \\
(\lambda_1 - \lambda_2)L_x^{\sigma}L_y^{\sigma} & \lambda_2(L_x^{\sigma})^2 + \lambda_1(L_y^{\sigma})^2
\end{pmatrix}$$

- then:

$$a = \frac{(\lambda_1(L_x^{\sigma})^2 + \lambda_2(L_y^{\sigma})^2)}{\|\nabla L^{\sigma}\|^2}$$

$$b = \frac{(\lambda_1 - \lambda_2)L_x^{\sigma}u_y^{\sigma}}{\|\nabla L^{\sigma}\|^2}$$

$$c = \frac{(\lambda_2(L_x^{\sigma})^2 + \lambda_1(L_y^{\sigma})^2)}{\|\nabla L^{\sigma}\|^2}$$
Figure 35: Comparaison Gaussian/P&M/Edge enhancing
Figure 36: Comparaison Gaussian/P&M/Edge enhancing
Edge Enhancing issue (1)

(a) Original image  
(b) Edge Enhancing
Edge Enhancing issue (2)

- Why?
- How are gradients oriented?

- In a same neighborhood, gradients may be opposed and compensate each other!
- How to fix that?
Part 1: linear scale space

Part 2: non linear scale spaces

Motivations

Digression: discretization of PDEs

Perona & Malik diffusion

Anisotropic diffusion

Edge Enhancing

Coherence Enhancing

Image restoration

To go further

APPENDIX
Coherence Enhancing [Weickert, 1998]

- Need to diffuse along edges but not in an oriented way
- Compute the (non oriented) direction of an edge
- Let’s define the tensor $S$ “local orientation”:

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} = \begin{pmatrix} L_x^s L_x^s \star G^\sigma & L_x^s L_y^s \star G^\sigma \\ L_x^s L_y^s \star G^\sigma & L_y^s L_y^s \star G^\sigma \end{pmatrix}$$

- Applied on $\nabla L$ and $-\nabla L$, $S$ gives the same value:

$$\begin{pmatrix} L_x^s L_x^s \star G^\sigma & L_x^s L_y^s \star G^\sigma \\ L_x^s L_y^s \star G^\sigma & L_y^s L_y^s \star G^\sigma \end{pmatrix}$$
Coherence Enhancing (2)

- The tensor $D$ is defined as follow:
  \[ D = R \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} R^T \]
  and $R$ is formed by the eigenvectors of $S$
- Eigenvectors of $S$ can be determined, $D$ writes:
  \[
  d_{11} = \frac{1}{2} \left( c_1 + c_2 + \frac{(c_1 - c_2)(s_{11} - s_{22})}{\alpha} \right)
  
  d_{22} = \frac{1}{2} \left( c_1 + c_2 - \frac{(c_1 - c_2)(s_{11} - s_{22})}{\alpha} \right)
  
  d_{12} = \frac{(c_2 - c_1)s_{12}}{\alpha}
  \]

with $\alpha = \sqrt{(s_{11} - s_{22})^2 + 4s_{12}^2}$
Coherence Enhancing (3)

- Weickert chooses for $c_1$ and $c_2$:

$$c_1 = \max(0.01, 1 - e^{-\frac{(\lambda_1 - \lambda_2)^2}{k^2}})$$
$$c_2 = 0.02$$

with $\lambda_1$ and $\lambda_2$ the eigenvalues of $S$:

$$\lambda_1 = \frac{1}{2} \left( s_{11} + s_{22} + \alpha \right)$$
$$\lambda_2 = \frac{1}{2} \left( s_{11} + s_{22} - \alpha \right)$$
Experimenting Coherence Enhancing (1)

(c) Original image

(d) Coherence Enhancing
Experimenting Coherence Enhancing (2)

- Artistic filtering?

Figure 37: “Route avec cyprès et Ciel étoilé”
Experimenting Coherence Enhancing (3)

- Artistic filtering?

**Figure 38:** Exacerbed Van Gogh
- color images:

Figure 39: independant diffusion on each channel
Figure 40: Only missing pixels are written and not read
Part 1: linear scale space

Part 2: non linear scale spaces
  Motivations
  Digression: discretization of PDEs
  Perona & Malik diffusion
  Anisotropic diffusion
  Edge Enhancing
  Coherence Enhancing
  **Image restoration**
  To go further

APPENDIX
Unblurring images?

- Gaussian blur $\Leftrightarrow$ linear and isotropic diffusion
- Question: given $L^n$ (blurred image), can one retrieve $L^0$?
- May be yes! discretization of heat equation using a retrograde scheme:

\[
\frac{L^{k+1} - L^k}{\Delta t} = \nabla^2 L^{k+1}
\]

\[
L^k = L^{k+1} - \Delta t \nabla^2 L^{k+1}
\]

(a) $\sigma = 1$  
(b) 5 itérations  
(c) 10 itérations

**Figure 41:** Backward scheme of heat equation
Gaussian unblurring (1)

(a) Blurred image
(b) Unblurring
Gaussian unblurring (2)

- The retrograde scheme is unstable:

**Figure 42:** Instability after 32 iterations
Example with a stronger blur:

**Figure 43:** $\sigma = 2$, 28 iterations
Gaussian unblurring (4)

• But not too much!

Figure 44: $\sigma = 3$: divergence after 15 iterations
Conclusion on Gaussian unblurring

- The retrograde scheme of heat equation is unconditionally unstable (easy to prove using Fourier analysis, exercise to do)
- In a same time, unblurring image implies to enhance / create edges: violation of the Causality Principle
- Rounding errors in the numerical retrograde scheme make appear noise and are amplified with increasing time iterations
- On the contrary, diffusion processes are, by nature, stable and regularizing: numerical errors or noise are smoothed out and eventually disappear
Let’s consider the following PDE:

\[
L_t = -\text{sign}(\nabla^2 L)\|\nabla L\| \quad t > 0
\]

\[
L(x, 0) = I(x)
\]

Properties:

- in a neighborhood of a local maxima \(x_0\):
  \(\nabla^2 u(x, t) < 0 \quad \forall x \in V(x_0)\):
  \[
  u_t = \|\nabla u\|
  \]

- in a neighborhood of a local minima \(x_0\):
  \(\nabla^2 u(x, t) > 0 \quad \forall x \in V(x_0)\):
  \[
  u_t = -\|\nabla u\|
  \]

These two cases are equivalent to morphological dilation or erosion

See [Osher and Rudin, 1990]
Image unblurring: shock filters (2)

- **Discretization:**

\[
L_i^{k+1} - L_i^k = \Delta t \left[ - \text{sign} (\nabla^2 L_i^k) \| \nabla L_i^k \| \right]
\]

- **Variation:**

\[
L_t = - \text{sign} \left( \frac{\partial L}{\partial \eta^2} \right) \| \nabla L \|
\]

with \( \eta = \frac{\nabla L}{\| \nabla u \|} \).

- [Alvarez and Mazorra, 1994]:

\[
L_t = - \text{sign} \left( \frac{\partial^2 v}{\partial \eta^2} \right) \| \nabla L \|
\]

with \( v = G_\sigma \ast L \) and \( \frac{\partial}{\partial \eta} \) derivative in the Gradient direction

- Anisotropic shock filter: [Weickert, 2003]
Part 1: linear scale space

Part 2: non linear scale spaces

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To go further

APPENDIX
To go further with non linear scale spaces (1)

- Unlike the continuous linear case, there are several nonlinear continuous multiscale representations.
- It is also possible to describe non linear spaces in an axiomatic way.
  The representation at scale $t$ is obtained with a nonlinear operator $T_t$ such as:
  - $L_t = T_t(I)$ (representation at scale $t$ of image $I$)
  - $T_t$ has a semi-group structure
  - $T_t$ respects the Causality Principle
  - $T_t$ is differentiable w.r.t. $t$
- Let’s denote $\partial T = \lim_{t \to 0} \frac{T_t(f) - f}{t}$, the family $(T_t(I))_{t>0}$ is solution of
  
  $L_t = \partial T(L)$
  $L(.,0) = I$
• Alvarez et al characterized the solutions of the previous PDE, see [Alvarez et al., 1993]

• Some interesting examples:
  • if $T$ is linear, the classic diffusion equation is retrieved
  • if $T$ commutes with a map $F(., t)$ monotone increasing, then solutions write:

$$u_t = |\nabla u| F \left( \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right), t \right)$$

this equation rules the dynamics of a snake evolving with a velocity depending only on the curvature of iso-value lines (implicit snake model)
To go further with non linear scale spaces (3)

- Some interesting examples:
  - Link with mathematical morphology: solutions of
    \[
    \begin{align*}
    u_t &= |\nabla u| \\
    u_t &= -|\nabla u|
    \end{align*}
    \]
    with \( u(x, 0) = I \) as initial condition are respectively dilatation and erosion of image \( I \) with a structuring element \( tB \) (\( B \) disc of radius 1)
    \( \Rightarrow \) successive dilatation/erosion of an image have the structure of a non linear scale space
  - Regularization: models formalized by a cost function to minimize and embedding a regularization term are equivalent to solve an Alvarez et al’s PDE

APPENDIX


**A Representation for Visual Information.**

*Pattern and search statistics.*
*Optimizing Methods in Statistics.*

*The structure of images.*
*Biological Cybernetics,* 50:363–370.

*Scale-Space Theory in Computer Vision.*

*Feature detection with automatic scale selection.*

Theory of edge detection.

  **Feature-oriented image enhancement using shock filters.**

  **Space scale and edge detection using anisotropic diffusion.**
  Article fondateur sur la diffusion non linéaire.

  **Edge and curve detection for visual scene analysis.**

  **Vector-valued image regularization with pdes: A common framework for different applications.**


Article fondateur sur les espaces d’échelles (1D).
Proof Eq. (5)

- determine \( \int e^{-x^2} \, dx \):

\[
\left( \int_{\mathbb{R}} e^{-x^2} \, dx \right)^2 = \left( \int_{\mathbb{R}} e^{-x^2} \, dx \right) \left( \int_{\mathbb{R}} e^{-y^2} \, dy \right) = \int_{\mathbb{R}^2} e^{-x^2-y^2} \, dxdy
\]

- Change of variable (polar coordinate): \( x = r \cos \theta \) et \( y = r \sin \theta \)

\[
\left( \int_{\mathbb{R}} e^{-x^2} \, dx \right)^2 = \int_{\mathbb{R}^+} \int_{0}^{2\pi} e^{-r^2} \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} \, drd\theta
\]

\[
= \int_{0}^{2\pi} \int_{0}^{+\infty} re^{-r^2} \, drd\theta = \int_{0}^{2\pi} d\theta \int_{0}^{+\infty} re^{-r^2} \, dr
\]

\[
= 2\pi \left[ -\frac{1}{2} e^{-r^2} \right]_{r=0}^{r=+\infty} = \pi
\]
Unicity of Gaussian kernel (1)

- Let \( L^{(t)}(x) = L(x, t) \)

- In Fourier space, equation (11) writes

\[
\hat{L}^{(t)}(w) = \hat{h}_t * \hat{I}(w) = \hat{h}_t(w) \hat{I}(w)
\]

\[
\frac{\hat{L}^{(t)}(w)}{\hat{I}(w)} = \hat{h}_t(w)
\]

with \( w = (u, v) \) coordinate in Fourier space

- Parameter \( w \) is a frequency, then \( w^{-1} \) is a period (a length)

- Parameter \( \sqrt{t} \) measures a scale (a length)

- Two lengths, then \( w \sqrt{t} \) is without dimension

- Same for \( \frac{\hat{L}^{(t)}(w)}{\hat{I}(w)} \)

- Pi theorem \( \Rightarrow \) one can write \( \hat{h}_t(w) = \hat{H}(\sqrt{tw}) \)
Unicity of Gaussian kernel (2)

- we then have: \( \frac{\tilde{L}(t)(w)}{L(w)} = \tilde{H}(w\sqrt{t}) \)

- if \( L = L^{(0)} \) then \( \tilde{H}(0) = 1. \)

- Semi-group: \( \tilde{h}_t(w) = \tilde{h}_{t_1 + t_2}(w) \) then:
  \[
  \tilde{H}(w\sqrt{t_1})\tilde{H}(w\sqrt{t_2}) = \tilde{H}(w\sqrt{t_1 + t_2})
  \]

- Let \( \tilde{H}(w^T w) = \tilde{H}(w), \ w \in \mathbb{R}^2 \), we have:

  \[
  \tilde{H}((w\sqrt{t_1})^T(w\sqrt{t_1})) \times \tilde{H}((w\sqrt{t_2})^T(w\sqrt{t_2})) = \tilde{H}((w\sqrt{t_1 + t_2})^T(w\sqrt{t_1 + t_2}))
  \]

  \[
  \tilde{H}(t_1w^Tw)\tilde{H}(t_2w^Tw) = \tilde{H}((t_1 + t_2)w^Tw)
  \]

  \[
  \tilde{H}(v_1)\tilde{H}(v_2) = \tilde{H}(v_1 + v_2)
  \]

  with \( v_1 = t_1w^Tw \) and \( v_2 = t_2w^Tw \)
Unicity of Gaussian kernel (3)

- Conclusion:
  \[
  \begin{cases}
  \hat{H}(0) = 1 \\
  \hat{H}(v_1)\hat{H}(v_2) = \hat{H}(v_1 + v_2)
  \end{cases}
  \]

- It’s the definition of the exponential function \((u \mapsto e^{\alpha u})\)

- Then: \(\hat{H}(w) = e^{\alpha t w^T w}\)

- We choose \(\alpha < 0\) in order to have \(\lim_{\infty} \hat{H} = 0\)

- The Fourier transform of an Gaussian function is a Gaussian function (with \(\alpha > 0\))
  \[
  \int_{\mathbb{R}^2} e^{-\frac{x^T x}{\alpha t}} e^{-iwx} dx = e^{-\alpha t w^T w}
  \]

- Normalization: \(\int_{\mathbb{R}} e^{-x^2/(2t)} dx = \sqrt{2\pi t}\), finally we choose \(\alpha = \frac{1}{2}\)

- Full proof available in the Lindeberg’s book [Lindeberg, 1994]