TADI - Wavelets - Tutorial works

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Fourier Analysis

Reminder:

$$
\begin{aligned}
\cos(p) + \cos(q) &= 2\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right) \\
\cos(p) - \cos(q) &= -2\sin\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right) \\
\sin(p) + \sin(q) &= 2\sin\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right) \\
\sin(p) - \sin(q) &= 2\sin\left(\frac{p-q}{2}\right)\cos\left(\frac{p+q}{2}\right) \\
\cos(x) &= \frac{e^{ix} + e^{-ix}}{2} \\
\sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} \\
e^{inx} &= \cos(nx) + i\sin(nx)
\end{aligned}
$$

$$
\begin{aligned}
\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\
\cos(a-b) &= \cos(a)\cos(b) + \sin(a)\sin(b) \\
\sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\
\sin(a-b) &= \sin(a)\cos(b) - \cos(a)\sin(b) \\
2\cos(a)\cos(b) &= \cos(a+b) + \cos(a-b) \\
2\sin(a)\sin(b) &= \cos(a-b) - \cos(a+b) \\
2\sin(a)\cos(b) &= \sin(a+b) + \sin(a-b) \\
X(f) : f \mapsto X(f) &= \int_{-\infty}^{+\infty} x(t)\, e^{-i2\pi f t}\, dt \\
\text{int. by part : } \int u'v &= [uv] - \int uv'
\end{aligned}
$$

Exercise 1: Fourier Series

1. Let E be the vector space of continuous functions defined on $[-\pi, \pi[$ valued in $\mathbb{C}$, with associated scalar product:
$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)\bar{g}(t)dt$$
   - Prove that the family $\mathcal{F}$ defined by $\{\cos(nx) \mid n \in \mathbb{N}\} \cup \{\sin(nx) \mid n \in \mathbb{N}^\star\}$ is orthogonal.
   - Is $\mathcal{F}$ a basis for F, the vector space of 2π periodic functions?
   - Propose an orthonormal basis for F.
2. Fourier Series: all 2π periodic function writes:
$$
\begin{aligned}
f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{+\infty} a_n \cos(nx) + b_n \sin(nx) \\
&= \sum_{n\in\mathbb{Z}} c_n e^{inx}
\end{aligned}
$$
   Express coefficients c_n as function of a_n and b_n.
3. Prove that the family $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthogonal basis for $L^2([0, 2\pi])$.

Exercise 2: Fourier transform

1. Let $z(t) = x(t-\tau)$, prove that $Z(f) = e^{-2i\pi f\tau}X(f)$.
2. Prove that if x is pair (respectively impair), its Fourier transform is pair (respectively impair).
3. Let x be such as $\lim_{t\to\pm\infty} x(t) = 0$, let $y = x'$, prove that $Y(f) = 2i\pi f X(f)$.
4. Let $y(t) = t\,x(t)$, prove that $X'(f) = -2i\pi Y(f)$.

Exercise 3: Fourier transform of usual functions

1. $\text{Rect}\left(\frac{t}{T}\right)$, with $\text{Rect}(t) = \begin{cases} 1 & \text{si } |t| \leq \frac{1}{2} \\ 0 & \text{sinon} \end{cases}$ (Gate or Rectangular function)
2. $x(t) = e^{-\alpha|t|}$, $\alpha > 0$
3. $g(t) = e^{-b^2t^2}$,
   - prove that $g'(t) + 2b^2tg(t) = 0$
   - deduce that $G'(f) + \frac{2\pi^2}{b^2}G(f) = 0$
   - and that $G(f) = \frac{\sqrt{\pi}}{|b|}e^{-\frac{\pi^2f^2}{b^2}}$
4. $k(t) = e^{-\alpha t}\mathbb{1}_{t\geq 0}$, $\alpha > 0$
5. $z(t) = t\,\mathbb{1}_{t\in]-a,a[}$

Exercise 4: frequency resolution and windowing

Let us consider the Sine function $x(t) = \cos(2\pi f_0 t)$ and the Rectangular function $r(t) = \text{Rect}\left(\frac{t}{L}\right)$. We recall that $X(f) = \frac{1}{2}(\delta(f-f_0) + \delta(f+f_0))$.

1. Determine the Fourier transform of $z(t) = x(t)r(t)$.
2. What can we conclude about the frequency resolution?

Exercise 5: Short-time Fourier Transform

Reminder:

$$\text{TFF}(x)(f,b) = \int_{\mathbb{R}} x(t)\bar{w}(t-b)e^{-2i\pi ft}dt$$

Let's consider the following 1-D signal:

$$x(t) = \cos(2\pi f_1 t)\,\text{Rect}\left(\frac{t-T_1}{2T_1}\right) + \cos(4\pi f_1 t)\,\text{Rect}\left(\frac{t-3T_1}{2T_1}\right)$$

with $f_1 = \frac{1}{T_1}$

1. Draw the graph of signal $x(t)$.
2. Determine and represent the spectrum of x for various disjoint temporal windows of length respectively $\frac{4}{f_1}$ (1 window), $\frac{2}{f_1}$ (2 windows), $\frac{1}{f_1}$ (4 windows), $\frac{1}{2f_1}$ (8 windows).
3. What previous windows can separate in time and in frequency the components of x? What is the best compromise?

Haar Wavelets

Let's consider the vector space of square integrable fonctions defined on $\mathbb{R}$ and denoted $E = L^2(\mathbb{R})$. Let ϕ be the scale function defined on $\mathbb{R}$ by:

$$\phi(t) = \begin{cases} 1 & t \in [0, 1[\\ 0 & \end{cases}$$

and $\phi_k^j(t) = \sqrt{2^j}\phi(2^j t - k)$:

$$\phi_k^j(t) = \begin{cases} \sqrt{2^j} & t \in [\frac{k}{2^j}, \frac{k+1}{2^j}[\\ 0 & \end{cases}$$

Exercise 6: multiresolution analysis of E

1. Prove that ϕ is an admissible scale function to build a multiresolution analysis of E.
2. Describe V^0, V^j.

We recall the multiresolution analysis of a vector space E:

1. $\forall j \in \mathbb{Z} \quad V^j \subset V^{j+1}$
2. $\lim\limits_{j \to -\infty} V^j = \bigcap\limits_{j \in \mathbb{Z}} V^j = \emptyset$
3. $\lim\limits_{j \to +\infty} V^j = \bigcup\limits_{j \in \mathbb{Z}} V^j = E$
4. $\exists \phi$ such as $\{\phi(. - k)\}_{k \in \mathbb{Z}}$ is an orthornormal basis of V^0
5. $\forall j \in \mathbb{Z}, f \in V^j \Leftrightarrow f(2.) \in V^{j+1}$
6. (consequence) $\forall j, k \in \mathbb{Z}, f \in V^j \Leftrightarrow f(. - 2^j k)$

Exercise 7: orthonormal basis of Haar scale functions

Let's consider the space of square integrable functions defined on $[0, 1[$: $E = L^2([0, 1[)$. We apply a multiresolution analysis using ϕ, we have:

- V^0 the space of constant functions on $[0, 1[$ (dimension 1)
- V^1 the space of constant functions on $[0, \frac{1}{2}[$ and $[\frac{1}{2}, 1[$ (dimension 2)
- V^j the space of constant functions on the 2^j intervals $[\frac{k}{2^j}, \frac{k+1}{2^j}[$, $k = 0, \cdots, 2^j - 1$ (dimension 2^j)

1. Verify that the family of functions $\left(\phi_k^j\right)_{k \in \{0, \cdots, 2^j - 1\}}$ is an orthonormal basis of V^j.
2. Draw the graph of functions ϕ_0^1 et ϕ_1^1 (basis of V^1), and of functions ϕ_0^2, ϕ_1^2, ϕ_2^2, ϕ_3^2 (basis of V^2).

Exercise 8: orthonormal basis of Haar details functions

1. Determine the two elements ψ_0^1 and ψ_1^1 of the basis of the vector space W^1 such as $V^2 = V^1 \oplus W^1$.
   - Compression: express ϕ_0^1, ϕ_1^1, ψ_0^1 and ψ_1^1 as functions of $(\phi_k^2)_{k\in\{0,1,2,3\}}$
   - Uncompression: express $(\phi_k^2)_{k\in\{0,1,2,3\}}$ as functions of ϕ_0^1, ϕ_1^1, ψ_0^1 and ψ_1^1
   - Determine the function ψ such as $\psi_k^1(t) = \sqrt{2}\psi(2t-k)$

2. Generalization: deduce the definition of the 2^j Haar details functions $\left(\psi_k^j\right)_{k\in\{0,\cdots,2^j-1\}}$ as an orthonormal basis of spaces W^j such as $V^{j+1} = V^j \oplus W^j$.

   a) Compression: express $\left(\phi_k^j\right)$ and $\left(\psi_k^j\right)$ as function of $\left(\phi_k^{j+1}\right)$

   b) Uncompression: express $\left(\phi_k^{j+1}\right)$ as function of $\left(\phi_k^j\right)$ and $\left(\psi_k^j\right)$

   c) Compression: express coefficients $\left(s_k^j\right)$ et $\left(d_k^j\right)$ as fonction of coefficients $\left(s_k^{j+1}\right)$

   d) Uncompression: express coefficients $\left(s_k^{j+1}\right)$ as function of coefficients $\left(s_k^j\right)$ and $\left(d_k^j\right)$

   e) Why the relation between coefficients s and d is the same than between functions ϕ and ψ?

Exercise 9: representation of a signal in the Haar wavelet basis

Let S be the following discrete signal $[2, 4, 8, 12, 14, 0, 2, 1]$.

1. Project S in the space $V^0 \oplus W^0 \oplus W^1 \oplus W^2$.
2. Draw the signal obtained for each resolution level (i.e. projection of S in V^2, V^1 and V^0), and details coefficients (i.e. projection of S in W^2, W^1 and W^0)).
3. The module of the Discrete Fourier Transform of S is: $[9, 8, 11, 24, 44, 24, 11, 8]$ (rounded to the closest integer).

   Discuss the interpretation in terms of scale space of the signal in the Haar wavelet space and in the Fourier space.