TADI: Wavelets
Master IMA/DIGIT
Sorbonne Université

Dominique.Bereziat@lip6.fr

This lecture is derived from Nicolas Thome's one.

year 2024-2025

Content

Vector space (1)

- Field: $(\mathbb{K},+,\cdot)$ a set with two operations (internal composition laws, denoted $+$ and $\cdot$)
  In general and in this lecture $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) and such as $+$ is commutative ($\forall\lambda,\mu\in\mathbb{K},\lambda+\mu=\mu+\lambda$), 0 is the neutral element for $+$ and 1 for $\cdot$
  - internal law: $\forall x,y\in\mathbb{K},x+y\in\mathbb{K}$
  - neutral element: $\forall x\in\mathbb{K},x+0=x$
- Vector space: $(E,+,\cdot)$ is a vector space over the field $\mathbb{K}$ if:
  - $\mathbb{K}$ is a field (two internal composition laws also denoted $+$ and $\cdot$ by abuse of language)
  - $+$ is an internal commutative law on E: $E\times E\to E$ (vector addition)
  - $\cdot$ is an external law (left multiplication): $\mathbb{K}\times E\to E$ (also called multiplication by a scalar) such as:
    - $\cdot$ is distributive over $+$: $\forall\lambda\in\mathbb{K},\forall v,w\in E,\lambda\cdot(v+w)=\lambda\cdot v+\lambda\cdot w$
    - $+$ is distributive over $\cdot$: $\forall\lambda,\mu\in\mathbb{K},\forall v\in E,(\lambda+\mu)\cdot v=\lambda\cdot v+\mu\cdot v$
    - 1 is the left neutral element of $\cdot$: $\forall v\in E,1\cdot v=v$
  - An element v of E is a vector, in the remaining E is a vector space

Vector space (2)

- Vector subspace: $F \subset E$ is a vector subspace of E if:
  - $F \neq \emptyset$
  - $\forall(\lambda, v, w) \in \mathbb{K} \times F \times F, \lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w \in F$,
- In other words: F is stable for linear combination
- Example of vector spaces:
  - $(\mathbb{R}^n, +, \cdot), (\mathbb{R}^{\mathbb{N}}, +, \cdot)$
  - The set of continuous functions from $\mathbb{R}$ into $\mathbb{C}$ is an $\mathbb{C}-$vector space (it is of infinite dimension)
- Scalar product: (or dot product, or inner product) the operation, denoted $\langle ., . \rangle$, such as:

$$\begin{array}{ccc} E \times E & \rightarrow & \mathbb{R} \\ (v, w) & \mapsto & \langle v, w \rangle \end{array}$$

  is a scalar product if
  - bilinear (linear on left, linear on right)
  - symmetric: $\langle v, w \rangle = \langle w, v \rangle$
  - positive: $\langle v, v \rangle \geq 0$
  - definite: $\langle v, v \rangle = 0 \Rightarrow v = 0$
- Norm: the scalar product defines the norm $\|v\|^2 = \langle v, v \rangle$

Scalar product

- A fundamental operation: it allows two vectors to be compared, projecting one to another one
- Example of scalar product:
  - in $\mathbb{R}^n$: $v = (v_1, \cdots, v_n), w = (w_1, \cdots, w_n)$ and

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i \cdot w_i$$

  - for the set of complex summable (or integrable) functions on $\mathbb{R}$:

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t)\bar{g}(t)dt$$

- Euclidean space: a vector space with a scalar product
- Hilbert space: an Euclidean space of infinite dimension (space of functions)

Basis (1)

- A basis in E is a finite or countable (if E is of infinite dimension) set of vectors of E: $\mathcal{B} = \{b_1, \cdots, b_n, \cdots\}$ satisfying two conditions:
  - linear independence property (free family): no element of $\mathcal{B}$ is a linear combination of others elements of $\mathcal{B}$: $\lambda_1 b_1 + \cdots + \lambda_n b_n = 0 \Rightarrow \lambda_1 = \cdots = \lambda_n = 0$
  - spanning property (spanning family): $\forall v \in E, \exists \lambda_1, \cdots, \lambda_n, \cdots$ such as $v = \sum_i \lambda_i b_i$
- Orthogonal basis: $\langle b_i, b_j \rangle = 0 \quad \forall i \neq j$
- Orthonormal basis: $\langle b_i, b_j \rangle = 0 \quad \forall i \neq j$ and $\langle b_i, b_i \rangle = 1 \quad \forall i$

Basis (2)

- Example in the Cartesian plane with the usual scalar product
  - the set reduced to the canonical vector $\vec{i} = (1 \quad 0)$: linearly independent set
  - $\{\vec{i}, \vec{j}, \vec{i} + \vec{j}\}$: spanning set
  - $\{2\vec{i}, \vec{i} + \vec{j}\}$: basis
  - $\{2\vec{i}, \vec{j}\}$: orthogonal basis
  - $\{\vec{i}, \vec{j}\}$: orthonormal basis (canonical basis)
  - $\left(\frac{\vec{i}+\vec{j}}{\sqrt{2}}, \frac{\vec{i}-\vec{j}}{\sqrt{2}}\right)$: orthonormal basis
- Consequences (without formal proof)
  - with a basis or a spanning set, one can represent any vector as $v = \sum_i \lambda_i b_i$
  - a linearly independent set can not represent all the vectors: for example, impossible to represent $\vec{j}$ as a linear combination of $\vec{i}$ (they are orthogonal)

Basis (3)

- Other consequences
  - Redundancy: a spanning set which is not a basis is a redundant set: there are too many vectors (at least one)
  - Redundancy: the representation of a vector is no more unique. For example with the spanning set $\{\vec{i}, \vec{j}, \vec{i}+\vec{j}\}$ and the vector $2 \cdot \vec{i}+\vec{j}$, one can exhibit two different linear combinations:

$$\begin{aligned} 2 \cdot \vec{i}+\vec{j} &= 2 \cdot \vec{i}+1 \cdot \vec{j}+0 \cdot (\vec{i}+\vec{j}) \\ &= 1 \cdot \vec{i}+0 \cdot \vec{j}+1 \cdot (\vec{i}+\vec{j}) \end{aligned}$$

  - Non orthogonal basis: the representation is unique but the determination of coefficients λ_i is not easy. In general:

$$v = \sum_i \lambda_i b_i \neq \sum_i \langle v, b_i \rangle \, b_i$$

  - Orthogonal basis: we have $\langle b_i, b_j \rangle = 0, i \neq j$ and

$$v = \sum_i \left\langle v, \frac{b_i}{\|b_i\|} \right\rangle \frac{b_i}{\|b_i\|}$$

    determination of λ_i are direct with the scalar product.
- Use of an orthonormal basis simplifies calculus

Conclusion

- Goals of theses recalls? Find suitable spaces of representation. Then find adapted basis.
- A well known example: Fourier Series! The $T-$ periodic functions may write as:

$$x(t) = \sum_{n\in\mathbb{N}} a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\left(\frac{2\pi nt}{T}\right) dt \quad b_n = \frac{2}{T}\int_0^T x(t)\sin\left(\frac{2\pi nt}{T}\right) dt$$

- Alternative writing:

$$x(t) = \sum_{k\in\mathbb{Z}} c_k e^{\frac{2i\pi kt}{T}} \quad (1)$$

$$c_k = \frac{1}{T}\int_0^T x(t) e^{\frac{-2i\pi kt}{T}} dt \quad (2)$$

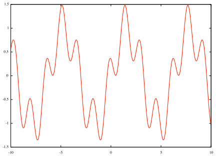
Here, we recognize the scalar product of a functional space: $c_k = \left\langle x, e^{\frac{2i\pi kt}{T}} \right\rangle$ and an orthonormal basis: $\{\phi_k\}_{k\in\mathbb{Z}}$ with $\phi_k(t) = e^{\frac{2i\pi kt}{T}}$

Content

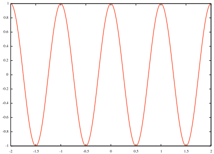
Fourier Series (1)

- Representation of the periodic functions
- Coefficient c_k are called Fourier coefficients
- The periodic function f is represented by the countable sequence $(c_k)_{k \in \mathbb{Z}}$
- Graphical interpretation:

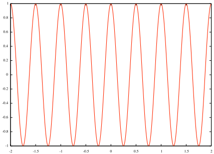
Given the following periodic signal:



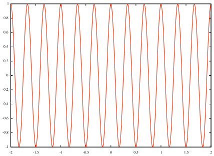
We have 8 non null Fourier coefficients: $c_{k_i} = c_{-k_i}, i = 1, \cdots, 4$ describing the 4 modes (pure frequencies) of this signal



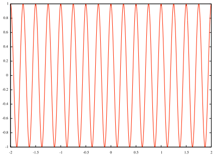
$$c_{k_1} = \left\langle x, e^{\frac{2i\pi k_1 t}{T}} \right\rangle$$



$$c_{k_2} = \left\langle x, e^{\frac{2i\pi k_2 t}{T}} \right\rangle$$



$$c_{k_3} = \left\langle x, e^{\frac{2i\pi k_3 t}{T}} \right\rangle$$



$$c_{k_4} = \left\langle x, e^{\frac{2i\pi k_4 t}{T}} \right\rangle$$

Fourier Series (2)

- Remark:
  - x even function $\Rightarrow c_k = c_{-k}$
  - x odd function $\Rightarrow c_k = -c_{-k}$

  On the previous example: linear combination of 4 cosine functions with various frequencies $\Rightarrow$ even function.
- Exercises:
  - show that the set $\{e^{\frac{2i\pi kt}{T}}\}_{k\in\mathbb{Z}}$ is an orthonormal basis
  - determine the Fourier coefficients of the function $t \mapsto \cos(2\pi \frac{t}{T})$
  - determine the Fourier coefficients of the Sawtooth wave (use a integration by parts to determine the integral of $t \mapsto te^{-2i\pi \frac{kt}{T}}$)
- See also: BIMA lecture on Fourier Transform

Fourier Transform (1)

Definition

- Applied on non-periodic function, the Fourier Series formulae does not work: $T = +\infty$ and $e^{2i\pi k\frac{t}{T}} = 1$, not a basis
- Extension to non-periodic functions: the Fourier Transform defined by

$$X(f) = \int_{\mathbb{R}} x(t)e^{-2i\pi ft}dt, f \in \mathbb{R}$$

- x must be an integrable function[1]. X is a continuous function on $\mathbb{C}$ and is an element of a vector space:
  - with the scalar product $\langle f, g\rangle = \int_{\mathbb{R}} f(t)\bar{g}(t)dt$
  - with the orthonormal basis: $\left\{t \mapsto e^{2i\pi ft}\right\}_{f\in\mathbb{R}}$, an element of the basis is the function $t \mapsto e^{2i\pi ft}$ indexed by the real parameter f

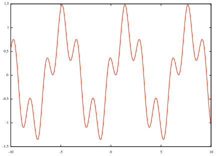
[1] f belongs to $L^2(\mathbb{R})$ space

Fourier Transform (2)

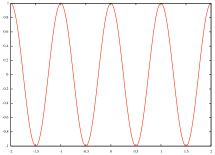
Graphical interpretation

- Same interpretation as the Fourier Series but on a continuous range of frequency

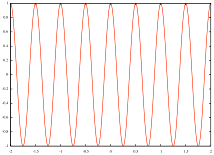
Given the following signal



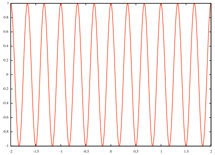
8 non null values for the Fourier transform: $X(f_i) = X(-f_i), i = 1, \cdots, 4$ describing the 4 modes of this signal



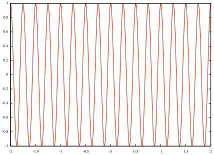
$X(f_1) = \left\langle x, e^{2i\pi f_1 t} \right\rangle$



$X(f_2) = \left\langle x, e^{2i\pi f_2 t} \right\rangle$



$X(f_3) = \left\langle x, e^{2i\pi f_3 t} \right\rangle$



$X(f_4) = \left\langle x, e^{2i\pi f_4 t} \right\rangle$

Fourier Transform (3)

Interpretation, reconstruction

- Interpretation:
  - magnitude: $|X(f)| = \sqrt{X(f)\bar{X}(f)}$, or spectral amplitude, gives the quantity of "pure" frequency f available in the signal x
  - phase: $\phi(f) = \arctan\left(\frac{\Re(X(j))}{\Im(X(f))}\right)$, gives the shift with the basis function $e^{2i\pi ft}$
  - fundamental or null frequency, $f = 0$, is the integral of the signal:

$$X(0) = \int_{\mathbb{R}} x(t)dt$$

- As with Fourier Series, reconstruction is possible:

$$x(t) = \int_{\mathbb{R}} X(f)e^{2i\pi ft}dt$$

FS versus FT

Fourier Series	Fourier Transform
x T-periodic functions	x integrable function
$c_k = \frac{1}{T}\int_0^T x(t)e^{-2i\pi\frac{k}{T}t}dt$	$X(f) = \int_{\mathbb{R}} x(t)e^{-2i\pi ft}dt$
$k \in \mathbb{Z}, c_k \in \mathbb{C}$	$X : \mathbb{R} \to \mathbb{C}$
$x(t) = \sum_{k\in\mathbb{Z}} c_k e^{2i\pi\frac{k}{T}t}$	$x(t) = \int_{\mathbb{R}} X(f)e^{2i\pi ft}df$

- To summary:
  - Fourier Series: periodic functions, countable orthonormal basis $\left(e^{2i\pi\frac{k}{T}t}\right)_{k\in\mathbb{Z}}$
  - Fourier Transform: integrable functions, uncountable orthornormal basis $\left(e^{2i\pi ft}\right)_{f\in\mathbb{R}}$

2-D Fourier Transform (1)

- An image is a non stationary function with a compact support, then is a non periodic function, Fourier Series are not suitable
- The 2-D Fourier Transform (for images) is built by separability:

$$\begin{aligned} X(f,g) &= \int_{\mathbb{R}}\int_{\mathbb{R}} x(t,u)e^{-2i\pi(ft+gu)}dtdu & (3) \\ &= \int_{\mathbb{R}}\left\{\int_{\mathbb{R}} x(t,u)e^{-2i\pi ft}dt\right\}e^{-2i\pi gu}du & (4) \end{aligned}$$

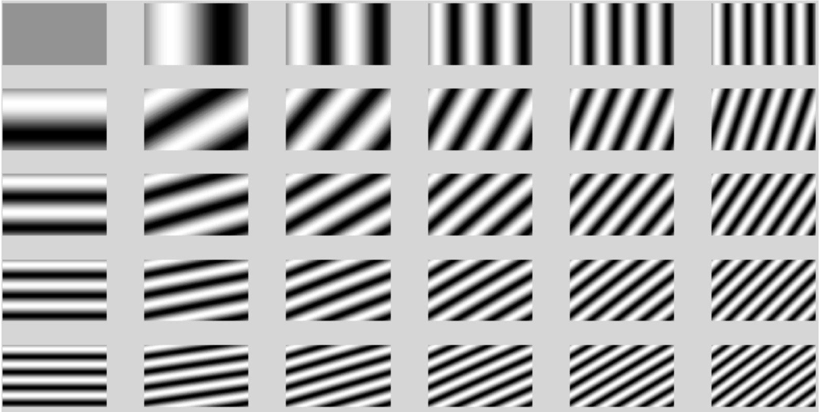
- $X:\mathbb{R}^2 \to \mathbb{C}$, (f,g) is a couple of vertical and horizontal frequencies
  - module of X (amplitude spectrum): $\sqrt{X\bar{X}}$, gives the amount of the element basis contained in signal x
    - basis: complex sinusoid ($(f,g) \mapsto e^{2\pi(ft+gu)}$)
  - phase of X: gives the phase change between signal x and the element basis
- Signal x can be reconstructed from its spectrum X with the inverse Fourier transform:

$$x(t,u) = \iint_{\mathbb{R}^2} X(t,u)e^{2i\pi(ft+gu)}dfdg$$

2-D Fourier Transform (2)

Inverse Fourier transform: any image is a linear combinaision of basis images

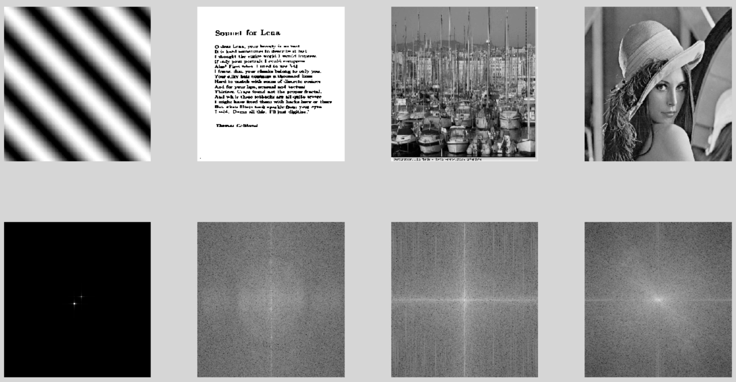
- an element of the basis, $(t, u) \mapsto \phi_{f,g}(t, u) = e^{2i\pi(ft+gu)}$, is an image!



2-D Fourier Transform (3)

Exemple sur des images

- module of spectrum: localize low and high frequencies, determine predominant orientations



Fourier transform: some mathematical tools (1)

Property (1-D or 2-D)

- linearity: $TF(\alpha x + \beta y) = \alpha X + \beta Y$
- scaling:

$$\begin{aligned} y(t) &= x(\alpha t) \\ Y(f) &= \frac{1}{\alpha} X\left(\frac{f}{\alpha}\right) \end{aligned}$$

- shift:

$$\begin{aligned} y(t) &= x(t - t_0) \\ Y(f) &= e^{-2i\pi f t_0} X(f) \\ |Y(f)| &= |X(f)| \end{aligned}$$

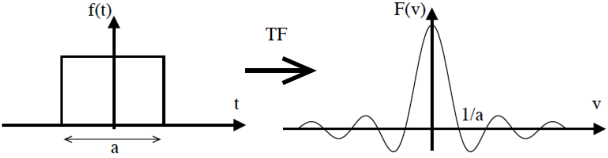
- rotation (for 2-D FT):

$$\begin{aligned} y(t, u) &= x(t \cos\theta + u \sin\theta, -t \sin\theta + u \cos\theta) \\ Y(f, g) &= X(f \cos\theta + g \sin\theta, -f \sin\theta + g \cos\theta) \end{aligned}$$

Fourier transform: some mathematical tools (2)

Fourier transform of some usual 1-D functions

- Rectangle function: $\text{Rect}(t) = \begin{cases} 1 & \text{si} \quad |t| \leq \frac{1}{2} \\ 0 & \text{sinon} \end{cases}$
- $TF[t \mapsto \text{Rect}\left(\frac{t}{a}\right)](f) = \int_{-a/2}^{a/2} e^{-2i\pi ft} dt = a\frac{\sin(\pi af)}{\pi af} = a\,\text{sinc}(\pi af)$

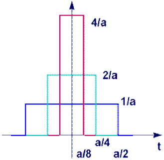


- Gaussian function:
  - $TF(t \mapsto e^{-b^2t^2})(f) = \frac{\sqrt{\pi}}{|b|} e^{-\frac{\pi^2 f^2}{b^2}}$, also a Gaussian function!
  - standard deviation in the frequency domain is inversely proportional to standard deviation in the time domain

Fourier transform: some mathematical tools (3)

Fourier transform of some usual 1-D functions

- Dirac delta function: δ. A generalized function (or distribution), formally defined by:
  - $\delta(x) = 0 \quad \forall x \neq 0$
  - $\int_{\mathbb{R}} \delta(x)dx = 1$
- Can be seen as the limit of normal function: $\delta(t) = \lim_{a \to 0} \frac{1}{a} \text{Rect}\left(\frac{t}{a}\right)$

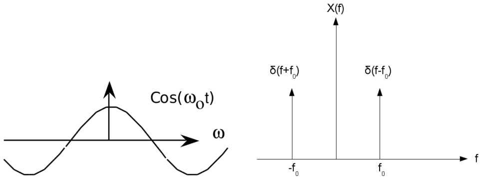


- Properties, for all function x
  - $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
  - $x \star \delta(t - t_0) = x(t - t_0)$, and then $x \star \delta(t) = x(t)$: δ neutral element of convolution
- Fourier transform:
  - $FT(t \mapsto \delta(t - t_0))(f) = e^{-2i\pi f t_0}$
  - $FT(t \mapsto e^{2i\pi f_0 t})(f) = \delta(f - f_0)$

Fourier transform: some mathematical tools (4)

Fourier transform of some usual 1-D functions

- Cosine function (Euler formulae):
  $FT[t \mapsto \cos(2\pi f_0 t)] = \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$



- Sine function: $FT[t \mapsto \sin(2\pi f_0 t)] = \frac{i}{2}(\delta(f - f_0) - \delta(f + f_0))$

Fourier transform: some mathematical tools (5)

Convolution theorem

- Recall, convolution:

$$z(t) = x \star y(t) = \int_{\mathbb{R}} x(t - t')y(t')dt'$$

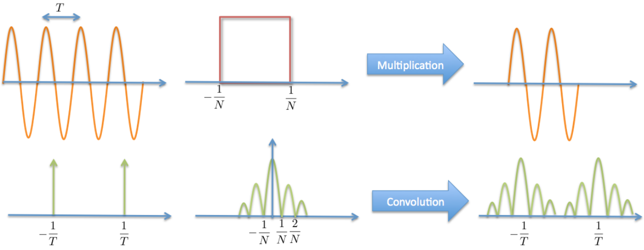
- Any linear filtering time invariant can be expressed by a convolution
- Convolution theorem:
  - if $z = x \star y$ then $Z = X \times Y$
  - if $z = x \times y$ then $Z = X \star Y$
- Important tool for calculation of Fourier transform! (see the next slide as an example)
- In 2-D (image), the convolution theorem still holds:

$$z(t, u) = x \star y(t, u) = \int_{\mathbb{R}} \int_{\mathbb{R}} x(t - t', u - u')y(t', u')dt'du'$$

- Consequence: filtering in the frequency domain is strictly equivalent to convolution in time (space) domain

Digitization and discrete Fourier transform (1)

- Practically: we analyze discrete signals and not real functions. A discrete tool is needed: the Discrete Fourier Transform (DFT)
- Formalization:
  1. the signal to analyze is windowed to obtain a bounded support function:
     - $x_L(t) = x(t)\,\mathrm{Rect}(t/L)$
     - FT: $X_L(f) = L\ X \star \mathrm{sinc}(\pi L f)$
- Example with a basic signal (cosine, pure frenquency)



Digitization and discrete Fourier transform (2)

- Practically: we analyze discrete signals and not real functions. A discrete tool is needed: the Discrete Fourier Transform (DFT)
- Formalization:
  1. the signal to analyze is windowed: $x(t) \Rightarrow x_L(t) = x(t)\,\mathrm{Rect}(t/L)$
  2. the windowed signal is sampled: a measure of this signal is done each T_s time step ($f_s = \frac{1}{T_s}$ is the sampling frequency):
     - $x_s(t) = x_L(t) \sum_{k\in\mathbb{Z}} \delta(t - kT_s)$ ($\sum_k \delta(t - kT_s)$: Dirac comb or train impulse)
     - Due to the windowing and the sampling frequency, we have $N = L/T_s$ measures
     - Fourier transform: $X_s(f) = X_L \star \sum_{k\in\mathbb{Z}} \delta(f - k/T_s)$ (the Fourier transform of Dirac comb is a Dirac comb). Hence: $X_s(f) = \sum_{k\in\mathbb{Z}} X_L(f - k/T_s)$

$\Rightarrow$ Sampling implies a periodic spectrum (of period $f_s = 1/T_s$)!

Digitization and discrete Fourier transform (3)

Sempling: Shannon theorem

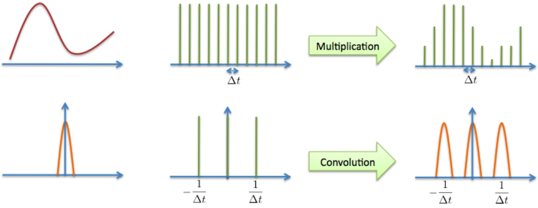


Figure: Sampling implies a periodic spectrum

- Let X be a bounded frequency support and let f_m be the maximal frequency of X:

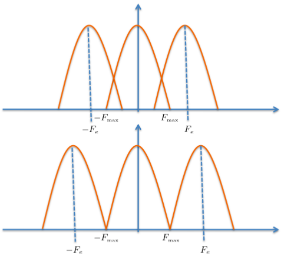
Theorem (Shannon)

If $f_s \geq 2f_m \Leftrightarrow T_s \leq \frac{1}{2}T_m$, *then the signal can be reconstructed without loss*

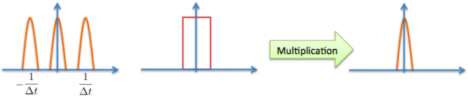
Digitization and discrete Fourier transform (4)

Échantillonnage: théorème de Shannon

- Spectrum overlapping if $f_m > f_s/2$ and limit case:



- Recontruction: X_L is truncated with a Rectangle function, then an inverse Fourier Transform is applied: Shannon interpolation formula



Digitization and discrete Fourier transform (5)

- Practically: we analyze discrete signals and not real functions. A discrete tool is needed: the Discrete Fourier Transform (DFT)
- Formalization:
  1. the signal to analyze is windowed:
     - $x_L(t) = x(t)\,\mathrm{Rect}(t/L)$
     - FT: $X_L(f) = L\ X \star \mathrm{sinc}(\pi L f)$
  2. the windowed signal is sampled:
     - $x_s(t) = x_L(t) \sum_{k\in\mathbb{Z}} \delta(t - kT_s)$
     - FT: $X_s(f) = \sum_{k\in\mathbb{Z}} X_L(f - k/T_s)$
  3. X_s is sampled at frequencies $f = \frac{k}{Nf_s}, k = 0 \cdots N-1$:
     - $\mathrm{DFT}(x)(k) = X_s\left(\frac{k}{Nf_s}\right), k = 0 \cdots N-1$
     - $\mathrm{DFT}(x)(k) = \sum_{n=0}^{N-1} x_s(n) e^{-2i\pi \frac{kn}{N}}, k = -\frac{N}{2} \cdots \frac{N}{2} - 1$
- Practically: we denote $x(k) = x(kT_s)$ as the k-th sample of signal x, and the Discrete Fourier transform is defined as:

$$\mathrm{DFT}(x)(k) = X(k) = \sum_{n=0}^{N-1} x(n) e^{-2i\pi \frac{kn}{N}}, k = -\frac{N}{2} \cdots \frac{N}{2} - 1 \qquad (5)$$

Discrete Fourier transform

Properties, and 2-D DFT

- DFT 2-D:

$$X(k,l) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} x(n,m) e^{-2i\pi\left(\frac{kn}{N} + \frac{lm}{M}\right)}$$

- The DFT has the same properties than the continuous Fourier transform:
  - linearity, translation and rotation of the signal/image
- Practically, DFT is used for filtering discrete signal/image in the frequency domain
- Inverse 2-D DFT:

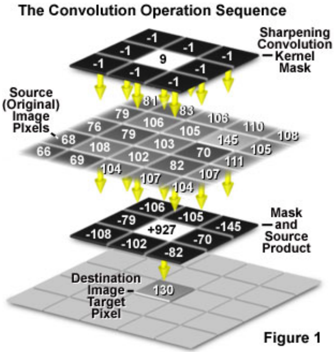
$$x(n,m) = \sum_{l=0}^{N-1} \sum_{k=0}^{M-1} X(k,l) e^{2i\pi\left(\frac{kn}{N} + \frac{lm}{M}\right)}$$

2-D discrete Fourier transform

Filtering in frequency domain vs time domain

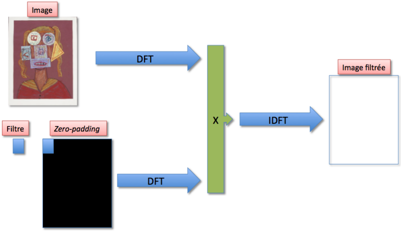
- Filtering in the time domain:

$$y(n,m) = x \star h(n,m)$$



- Filtering in the frequency domain:

$$y(n,m) = TFD^{-1}[X(u,v) \times H(u,v)]$$



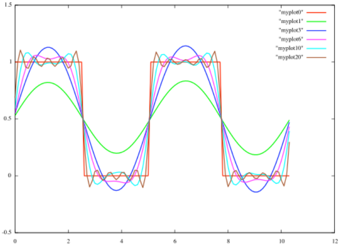
Filtering in the frequency domain

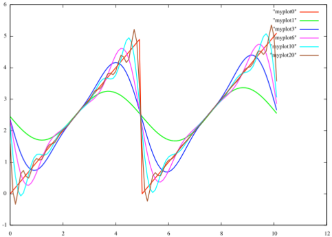
- Several types of filters:
  - low-pass: low frequencies are kept, high frequencies are attenuated
  - high-pass: low frequencies are attenuated, high frequencies are attenuated
  - band-pass: a range of frequencies is kept, others frequencies are attenuated: allow an multi-scale analysis (scale=size of structures)
- See BIMA course (https://www-master.ufr-info-p6.jussieu.fr/parcours/ima/bima/): lectures 3, 4, 5 and associated tutorial and practical works.

Content

FT: limitations, issues (1)

- Compression, denoising: impossible to correctly represent edges (non derivable functions): Gibbs ringing artifacts appear after removing highest frequencies





- Visible in JPEG compression for example

FT: limitations, issues (2)

- In the Fourier space, structure size and orientation can be measured but it is not possible to localize (translation invariant): a wave has a period (size), an orientation (in 2-D), a phase, but not a localization.
- Two ways to represent a signal:
  - representation in time (or spatial if image) domain:

    $$x(t) = \int_{\mathbb{R}} x(u)\delta(t-u)du$$

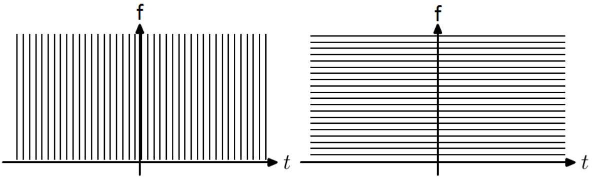
    => this basis localizes in time, but not in frequency (it can't see the size of structures)
  - representation in the frequency domain (inverse FT):

    $$x(t) = \int_{\mathbb{R}} X(f)e^{2i\pi ft}df$$

    => this basis localizes in frequency but not in time

FT: limitations, issues (3)

- Representation in time domain: null resolution in frequency, infinite resolution in time
- Representation frequency domain: infinite resolution in frequency, null resolution in time



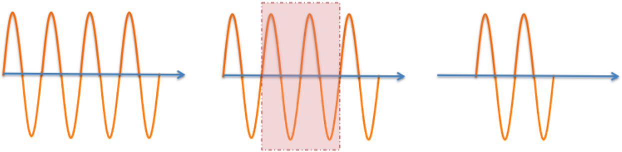
(a) DIRAC (b) FOURIER

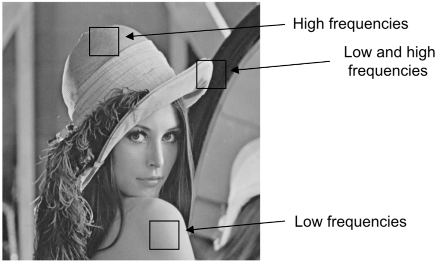
- Consider two signals:
  - $y(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t)$
  - $z(t) = \sin(2\pi f_1 t)u(t) + \sin(2\pi f_2 t)u(-t)$ with $u(t) = 1$ if $t > 0$ and 0 otherwise (Heavyside function)

  y and z has the same spectrum!
- Need to analyze the signal both in time and in frequency domains!

Short Time Fourier Transform (1)

- Principe: perform a Fourier analysis on a window.
  - first the signal is windowed, the window being localized in the time domain, second a Fourier Transform is applied
  - the STFT has two parameters:
    - a parameter of time localization
    - a parameter of frequency localization





- Other name: Windowed Fourier Transform

Short Time Fourier Transform (2)

- Definition:

$$STFT(x)(f,b) = X(f,b) = \int_{\mathbb{R}} x(t)\bar{w}(t-b)e^{-2i\pi ft}dt$$

with w an *admissible* window, i.e. $\int_{\mathbb{R}} |w(t)|^2 dt = 1$

- Examples for w: Rectangle function, Triangle function, Gaussian function, ...
- The family of functions $\phi_{f,b}(t) = w(t-b)e^{2i\pi ft}$ is spanning but redundant set (two parameters f and b)
  - STFT: $\phi_{f,b}(t) = w(t-b)e^{2i\pi ft}$: localization in frequency f and in time b
  - FT: $\phi_f(t) = e^{2i\pi ft}$: localization only in frequency
- Reconstruction is available if w is an admissible window:

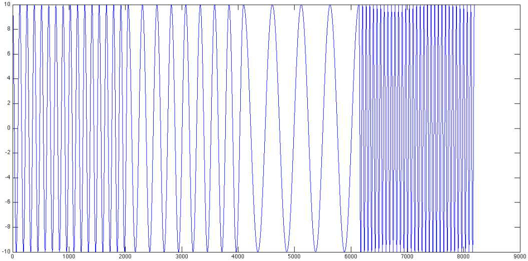
$$x(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} X(t,b)w(t-b)e^{2i\pi ft}dfdb$$

- Exercise: prove the reconstruction formula

Short Time Fourier Transform (3)

Example

- Time-varying frequency signal:

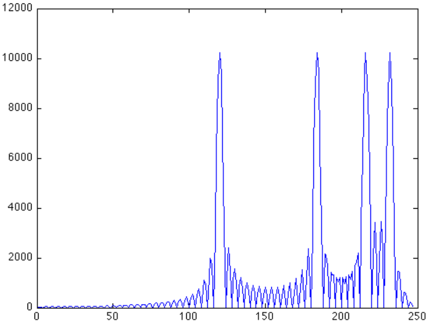


$$x(t) = \sum_{k=1}^{4} \cos(2\pi f_k t)\,\mathrm{Rect}\left(\frac{t - t_k}{w}\right)$$

Short Time Fourier Transform (4)

Exemple

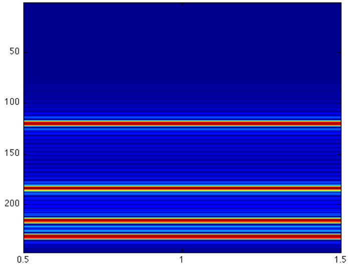
- Fourier transform of x: no localization in time!



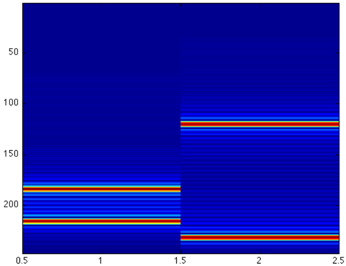
$$X(f) = \sum_{k=1}^{4} \frac{\delta(f - f_k) + \delta(f + f_k)}{2} \star e^{-2i\pi f t_k} \operatorname{sinc}(w\pi f)$$

Short Time Fourier Transform (5)

Example: representation time-frequency



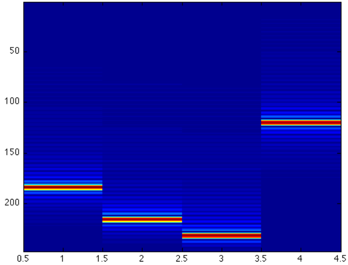
1 window: it is the standard Fourier Transform, so no localization in time



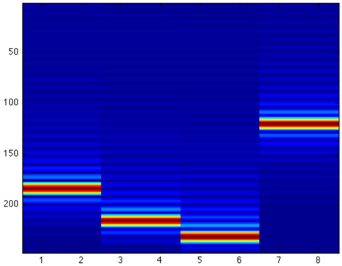
2 windows: gain in time resolution

Short Time Fourier Transform (6)

Example: representation time-frequency



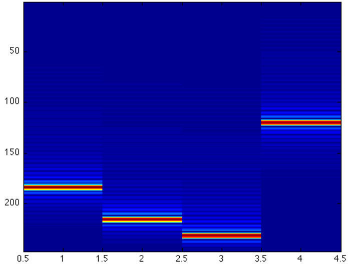
4 windows: gain in time resolution



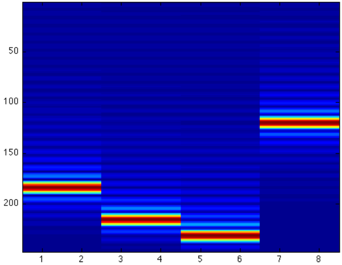
8 windows: loss of frequency localization and then frequency resolution! why?

Short Time Fourier Transform (6)

Example: representation time-frequency



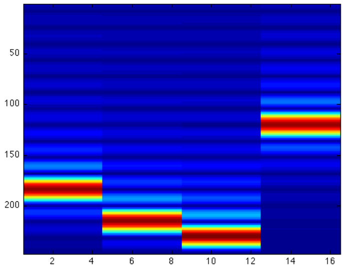
4 windows: gain in time resolution



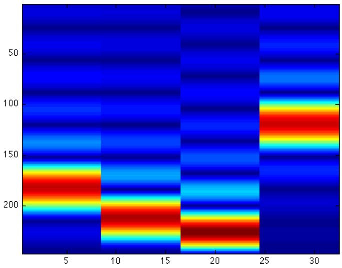
8 windows: loss of frequency localization and then frequency resolution! why? as the window becomes smaller, the FT (sinc) is lesser accurate

Short Time Fourier Transform (7)

Example: representation time-frequency



16 windows: loss of frequency resolution!



32 windows: loss of frequency resolution!

Short Time Fourier Transform (8)

Example: representation time-frequency

- Conclusion: there is an optimal configuration to analyze the x signal
  - with less than 4 windows: low time resolution but good frequency resolution
  - more than 4 windows: maximal time resolution, but low frequency resolution résolution fréquentielle moins bonne
  - 4 windows is the optimal in this case
- See Exercise 5 in tutorial works

Short Time Fourier Transform (9)

Limitations, issues

- Window length is a critical parameter:
  - must be the same order of value than the period of the signal to be analyzed
  - but not so large, because the time resolution will be degraded
- Let us formally define the time and frequency resolution of a x signal:
  - $< t > = \frac{1}{E}\int_{\mathbb{R}} t|x(t)|^2 dt$, $< f > = \frac{1}{E}\int_{\mathbb{R}} f|X(f)|^2 df$
    with $E = \int_{\mathbb{R}} |x(t)|^2 dt$
  - Time resolution (standard deviation, dispersion):

$$\sigma_t = \int_{\mathbb{R}} (t - < t >)^2 |x(t)|^2 dt$$

  - Frequency resolution (standard deviation, dispersion):

$$\sigma_f = \int_{\mathbb{R}} (f - < f >)^2 |X(f)|^2 df$$

- small standard deviation ⇒ high localization ⇒ high resolution

Heisenberg uncertainty principle

- A general principle apply to any waves (and more):
  - impossible to localize both in time and in frequency with a infinite precision a signal
  - time and frequency resolution are bounded: $\sigma_t \sigma_f \geq \frac{1}{4\pi}$

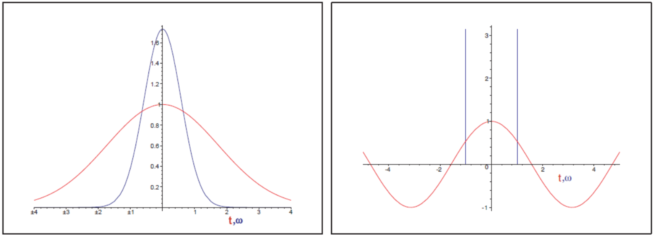
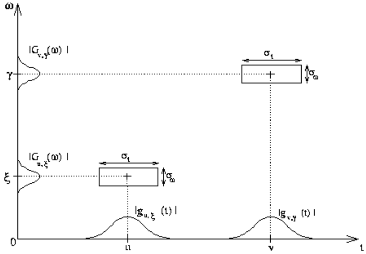


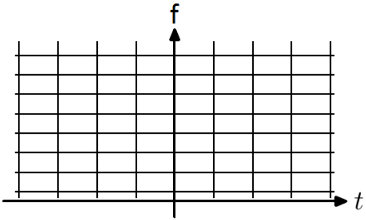
Figure: Left: Gaussian signal (red) and its spectrum, right: Cosine signal and its spectrum

  - The bound is reached with the Gaussian function!

Heisenberg boxes

1. Time and frequency resolution can be represented using the Heisenberg boxes:





2. Here: σ_t and σ_f are constant.
3. Too large window: impossible to analyze non stationary signals (loss of localization in time)
4. Too small window: loss of localization in frequency
5. Idea of wavelets: analyze in time and frequency more suitable (i.e. Heisenberg boxes of various size), and design of an orthonormal basis (STFT is not a basis)

Content

Continuous wavelet transform (CWT): definition

- $E = L^2(\mathbb{R})$ set of real function squared integrable (a vector space)
- Let $x \in E$ be a signal, the continuous wavelet transform is a function $(a, b) \mapsto g(a, b)$ defined by:

$$g(a,b) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} x(t)\bar{\psi}_{a,b}(t)dt = \langle x, \psi_{a,b} \rangle$$

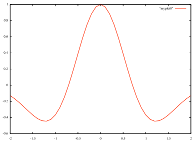
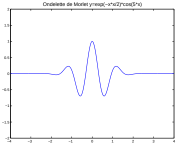
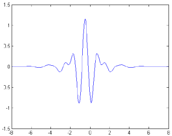
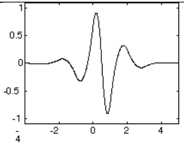
  such as $a \neq 0$ and:

$$\psi_{a,b} = \frac{1}{\sqrt{a}} \psi \left(\frac{t-b}{a} \right)$$

  where ψ is called mother wavelet
- Functions $\psi_{a,b}$ are translated/dilated version of ψ
- b: position (localization in time), a: scale (analog of the period of Fourier analysis)

Mother wavelet

- ψ must be *admissible*:
  - has a bounded support
  - is of mean null ($\int \psi = 0$)
  - be oscillating $|\psi| \neq \psi$
  - $\psi \in E$ (squared integrable)
  - $\psi(t) \in \mathbb{R}$ or $\mathbb{C}$
- Examples:

	
(a) Mexican hat: $\Psi(t) = (1 - t^2)e^{(-\frac{t^2}{2})}$	(b) Morlet wavelet $\Psi(t) = e^{iat} e^{\frac{-t^2}{2\sigma^2}}$
(c) Meyer wavelet	(d) Biorthogonal spline wavelet
	

CWT versus STFT

- Similarity:
  - Both are redundant analysis (projection onto redundant spanning families)
  - Both localize in time and in frequency domains:
    - STFT: $\phi_{f,b} = w(t-b)e^{2i\pi ft}$
    - CWT: $\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$
- Difference:
  - STFT: has a fixed resolution in time and in frequency (Heiseinberg boxes have the same size)
  - CWT: has a variable resolution in time and in frequency
- Interpretation for the CWT:
  - allow a multiscale analyze: the support in the time domain is more or less large (the mother wavelet is dilated at various size)
- Let $\sigma_t^{a,b}$ et $\sigma_f^{a,b}$ be the respective time and frequency resolution of $\psi_{a,b}$:
  - $\sigma_t^{a,b} = a\sigma_t^{1,0}$
  - $\sigma_f^{a,b} = \frac{1}{a}\sigma_f^{1,0}$

  with $\sigma_t^{1,0}$ and $\sigma_f^{1,0}$ the time and frequency resolution of mother wavelet ψ

Heisenberg boxes

- Recall: Heinsenberg incertitude principle, $\sigma_t \sigma_f \geq \frac{1}{4\pi}$, boxes have a minimal area

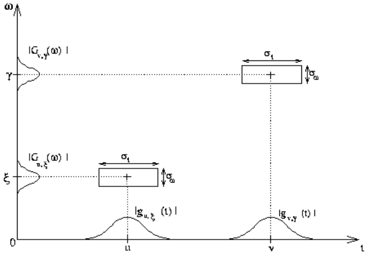


Figure: Heisenberg box of FT

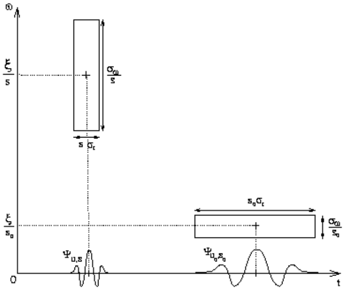


Figure: Heisenberg box of CWT

CWT: interpretation

Wavelet as a multi-scale analysis tool

- Findings:
  1. low frequencies are less localized in time: a low frequency signal has a long period and is almost stationary
  2. high frequencies are better localized in time (small period) and non stationary, their localization in time are important for analysis
- Wavelets: a frequency is analyzed at a suitable time resolution:
  1. low frequency (scale a is large): low time resolution, high frequency resolution
  2. high frequency (scale a is small): high time resolution, low frequency resolution

  There is a compromise between time and frequency resolution (Heisenberg)

Reconstruction

- Formally:

$$x(t) = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} a^{-2} g(a, b) \psi_{a,b}(t) dadb$$

  with

$$C_\psi = \int_0^{+\infty} \frac{|\psi(f)|^2}{f} df$$

- If $C_\psi < \infty$ (admissibility condition), reconstruction is possible
- The family is redundant: practically, reconstruction is costly, but:
  - a countable set of values for $(a, b) \mapsto g(a, b)$ is sufficient to reconstruct x,
  - practically, a continuous wavelet transform is not suitable for discrete signal: a discrete formulation of wavelet is requested

Content

Reducing redundancies: Dyadic wavelets

- The continuous wavelet transform is sampled using a *dyadic* position:
  - $a = 2^{-j}$
  - $b = k \times 2^{-j}, \quad k = 0, \cdots, 2^j - 1$
- $j \in \mathbb{N}$ is the time resolution (or representation scale)
- $\psi_{a,b}(t) = \sqrt{2^j}\psi(2^j t - k) = \psi_k^j(t)$ has a support of length 2^{-j} and a position at k
- For j fixed, $\psi_k^j(t)$ functions have disjoint and contiguous supports. Let ψ be a mother wavelet with support on $[0, 1]$:
  - $j = 0$: $k = 0$. Only one function for this scale, $\psi_0^0(t) = \psi(t)$
  - $j = 1$: $k = 0$ or 1. Two functions for this scale:
    - position 0: $\psi_0^1(t) = \sqrt{2}\psi(2t)$ with support on $[0, \frac{1}{2}]$
    - position 1: $\psi_1^1(t) = \sqrt{2}\psi(2t - 1)$ with support on $[\frac{1}{2}, 1]$
  - $j = 2$: $k = 0, 1, 2, 3$, 4 functions:
    - position 0: $\psi_0^1(t) = \sqrt{2}\psi(4t)$, support on $[0, \frac{1}{4}]$
    - position 1: $\psi_1^1(t) = \sqrt{2}\psi(4t - 1)$, support on $[\frac{1}{4}, \frac{1}{2}]$
    - position 2: $\psi_2^1(t) = \sqrt{2}\psi(4t - 2)$, support on $[\frac{1}{2}, \frac{3}{4}]$
    - position 3: $\psi_3^1(t) = \sqrt{2}\psi(4t - 3)$, support on $[\frac{3}{4}, 1]$
  - ...

Dyadic wavelets

- Redundancy is reduced: $(a, b) \in \mathbb{R}^2 \Rightarrow (j, k), j \in \mathbb{N}, 0 \leq k < 2^j$: countable family
- We obtain a discrete sequence of coefficients:

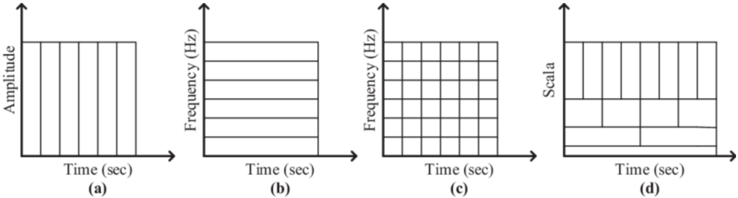
$$g_k^j = \left\langle x, \psi_k^j \right\rangle$$

- Reconstruction:

$$x(t) = \sum_{j \in \mathbb{N}} \sum_{k=0}^{j} g_k^j \psi_k^j(t)$$

- Remark: this transform applies on continuous signal (x is continuous as well the elements of the family, $t \mapsto \psi_k^j(t)$). We do not yet have a discrete transform.

Dyadic wavelets transform versus FT, STFT



(a) Localization in time domain
(b) Localization in frequency domain (FT)
(c) Localization in time and frequency domains (STFT)
(d) Localization in scale and time domains (dyadic wavelet)

Content

Multiresolution analysis (1)

Motivations

- Dyadic wavelets: the family is not redundant but the basis is not orthogonal (eg: $\left\langle \psi_k^j, \psi_{2k}^{j+1} \right\rangle \neq 0$)
- Multiresolution analysis: formalism to build wavelet orthornormal basis
- Principle: project the signal into nested vector subspaces

Multiresolution analysis (2)

Definition

- A multiresolution analysis of $E = L^2(\mathbb{R})$ is a sequence of subspaces $(V^j)_{j\in\mathbb{Z}}$ such as:
  1. information contained in resolution j is also contained in resolution $j+1$: $\forall j \in \mathbb{Z} \quad V^j \subset V^{j+1}$
  2. intersection of all V^j is empty: $\bigcap_{j\in\mathbb{Z}} V^j = \lim_{j\to-\infty} V^j = \emptyset$
  3. union of all V^j is E: $\bigcup_{j\in\mathbb{Z}} V^j = \lim_{j\to+\infty} V^j = E$
  4. resolution j derives from resolution $j+1$ by a dilation of factor 2: $\forall j \in \mathbb{Z} \quad f \in V^j \Leftrightarrow f(2.) \in V^{j+1}$
  5. it exists a function $\phi \in E$ such as the family $(\phi(.-k))_{k\in\mathbb{Z}}$ is an orthonormal basis in V^0
- Consequences:
  - from 4. and 5. it comes: $\forall k \in \mathbb{Z} \quad f \in V^j \Leftrightarrow f(.-k2^j) \in V^j$. In other words $(\phi(.-k2^j))_{k\in\mathbb{Z}}$ is a basis in V^j
  - from 3.: one can reconstruct a signal $x \in E$ from its projections into V^j
- ϕ is known as scaling function (or wavelet father)
- V^j are known as the approximation subspaces

Multiresolution analysis (3)

scaling function: one example

1. Consider $\phi(t) = 1$ on $[0, 1[$, null otherwise
2. This is Haar scaling function
3. What does V^0 represent?, V^j?

Multiresolution analysis (3)

scaling function: one example

1. Consider $\phi(t) = 1$ on $[0, 1[$, null otherwise
2. This is Haar scaling function
3. What does V^0 represent?, V^j?
   - $E = L^2(\mathbb{R})$, scalar product: $\langle f, g \rangle = \int_{\mathbb{R}} f(t)\bar{g}(t)dt$
   - suppose $\phi(t-k)$ is a basis in V^0 then if $f \in V^0$, $f(t) = \sum_{k\in\mathbb{Z}} \langle f, \phi(. - k) \rangle \, \phi(t-k) = \sum_k c_k \phi(t)$ with $c_k = \int_{\mathbb{R}} f(t)\bar{\phi}(t-k)dt = \int_k^{k+1} f(t)dt$
   - then V^0 is the space of functions constant on intervals $[k, k+1[$
   - and then V^1 is the set of functions constant on intervals $[k/2, (k+1)/2[$ if condition 4 holds.
   - and then V^j is the set of functions constants on intervals $[2^{-j}k, 2^{-j}(k+1)[$

Multiresolution analysis (3)

scaling function: one example

1. Consider $\phi(t) = 1$ on $[0, 1[$, null otherwise
2. This is Haar scaling function
3. What does V^0 represent?, V^j?
4. Is Haar scaling function admissible to perform a multiresolution analysis of $E = L^2(\mathbb{R})$?

Multiresolution analysis (3)

scaling function: one example

1. Consider $\phi(t) = 1$ on $[0, 1[$, null otherwise
2. This is Haar scaling function
3. What does V^0 represent?, V^j?
4. Is Haar scaling function admissible to perform a multiresolution analysis of $E = L^2(\mathbb{R})$?
   - condition 5. is true: $\phi(. - k)$ is an orthonormal basis in V^0, easy to verify
   - condition 1. ($V^j \subset V^{j+1}$): if $f \in V^j$ then f constant on intervals $[2^{-j}k, 2^{-j}(k+1)[$, and also constant on intervals $[2^{-(j+1)}k, 2^{-(j+1)}(k+1)[$ and we conclude $f \in V^{j+1}$
   - conditions 2. and 3. intuitively: integral of a function may be approximated by piecewise constant functions (integral definition in sense of Riemann)
   - condition 4. (transition j to $j+1$): similar proof than for condition 1, $f(2.)$ is a dilatation of f by a factor 2, then $f(2.) \in V^{j+1}$
5. Haar scaling function is an admissible solution for a multiresolution analysis of E (see Ex 6 tutorial works)

Multiresolution analysis (4)

Projection into V^j

- Let ϕ be an admissible scaling function in $E = L^2(\mathbb{R})$
- Let's define: $\phi_k^j(t) = \sqrt{2^j}\phi(2^j t - k)$, then:
  - $\left(\phi_k^j\right)_{k\in\mathbb{Z}}$ is an orthonormal basis in V^j
    - derives from conditions 4. and 5.
- Given $x \in E$, its projection into V^j is:

$$x^j(t) = (P_j x)(t) = \sum_k s_k^j \phi_k^j(t)$$

with:

$$s_k^j = \left\langle x, \phi_k^j \right\rangle_{V^j} = \int_{\mathbb{R}} \sqrt{2^j} x(t)\phi(2^j t - k)dt$$

we recognize a scalar product for V^j

- s_k^j are the approximation coefficients at resolution j
- Subspaces V^j are dyadic spaces

Multiresolution analysis (5)

Complementary subspaces (1)

- Last step: obtain an orthonormal basis
- Fundamental idea: as $V^j \subset V^{j+1}$ then

$$\exists W^j \text{ such as } V^{j+1} = V^j \oplus W^j$$

  W^j is known as the details subspace for resolution j
- W^j is a complementary subspace orthogonal to V^j in V^{j+1}
- We call wavelets (or details functions) the set of functions $\left(\psi_k^j\right)_{k\in\mathbb{Z}}$ spanning W^j and pairwise orthogonal
- Having an orthonormal basis in V^j and in W^j, we have an orthonormal basis in V^{j+1} : $\left(\phi_k^j\right)_{k\in\mathbb{Z}} \bigcup \left(\psi_k^j\right)_{k\in\mathbb{Z}}$ and

$$x^{j+1}(t) = \underbrace{\sum_{k\in\mathbb{Z}} s_k^j \phi_k^j(t)}_{\text{projection into } V^j} + \underbrace{\sum_{k\in\mathbb{Z}} d_k^j \psi_k^j(t)}_{\text{projection into } W^j}$$

- $d_k^j = \left\langle x, \psi_k^j \right\rangle$ are known as the details coefficients

Multiresolution analysis (5)

Complementary subspaces (2)

- Recursively we have:

$$\begin{aligned} V^{j+1} &= V^j \oplus W^j = V^{j-1} \oplus W^{j-1} \oplus W^j \\ &= V^0 \oplus W^0 \oplus W^1 \oplus \cdots \oplus W^{j-1} \oplus W^j \\ x^{j+1}(t) &= \sum_k s_k^0 \phi_k^0(t) + \sum_{i=0}^{j} \sum_k d_k^i \psi_k^i(t) \end{aligned}$$

- Basis in V^{j+1} contains:
  - that of V^0
  - that of W^0, W^1, up to W^j
- $j \to +\infty$:
  - $E = L^2(\mathbb{R}) = V^0 \bigoplus_{i=0}^{+\infty} W^j$
  - $x(t) = \sum_k s_k^0 \phi_k^0(t) + \sum_{i=0}^{+\infty} \sum_k d_k^i \psi_k^i(t)$

Multiresolution analysis (5)

Complementary subspaces (3)

- Subspaces V^j are also nested when $j < 0$: $\cdots \subset V^{-1} \subset V^0$
- Then:

$$
\begin{aligned}
E &= V^0 \bigoplus_{i=0}^{+\infty} W^j \\
&= V^{-1} \oplus W^{-1} \bigoplus_{i=0}^{+\infty} W^j \\
&= V^{-j} \oplus W^{-j} \oplus \cdots \oplus W^{-1} \bigoplus_{i=0}^{+\infty} W^j \\
&= \bigoplus_{j=-\infty}^{+\infty} W^j \\
x(t) &= \sum_{j=-\infty}^{+\infty} \sum_k d_k^j \psi_k^j(t)
\end{aligned}
$$

Multiresolution analysis (6)

Conclusion

- The multiresolution analysis allows to build a basis of orthogonal wavelets $\left(\psi_k^j\right)$
- Subspaces V^j have a dyadic basis $\left(\phi_k^j\right)$ derived from the scaling function ϕ (also named father wavelet): $\phi_k^j(t) = \sqrt{2^j}\phi(2^j t - k)$
- Complementary subspaces W^j also have a dyadic basis derived from the mother wavelet ψ: $\psi_k^j(t) = \sqrt{2^j}\psi(2^j t - k)$
- Issue: choose ψ

Content

Haar wavelet (1)

- $E = L^2([0,1[)$, $x : E \to \mathbb{R}$
- Scaling function (Haar):

$$\phi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Bases of subspaces V^j: $\phi_k^j(t) = \sqrt{2^j}\phi(2^j t - k)$:

$$\phi_k^j(t) = \begin{cases} \sqrt{2^j} & \frac{k}{2^j} \leq t < \frac{k+1}{2^j} \\ 0 & \text{otherwise} \end{cases}$$

- We conclude that:
  - V^0 is the set of constant functions on $[0,1[$, spanned by ϕ_0^0
  - V^1 is the set of constant functions on $[0, \frac{1}{2}[$ and $[\frac{1}{2}, 1[$, spanned by ϕ_0^1 and ϕ_1^1
  - V^j is the set of constant functions on $[\frac{k}{2^j}, \frac{k+1}{2^j}[$, $k = 0, \cdots, 2^j - 1$
  - V^{-1} do not make sense

Haar wavelet (2)

- The mother wavelet can be chosen as:

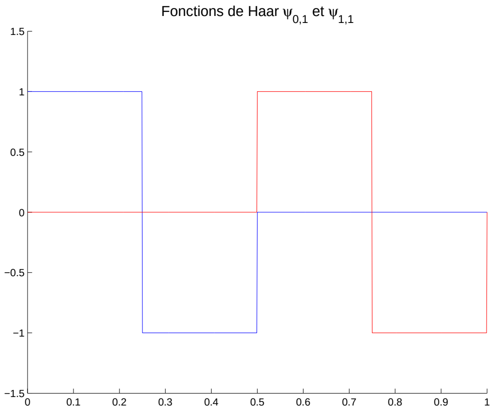
$$\psi(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

- And for other wavelets: $\psi_k^j(t) = \sqrt{2^j}\psi(2^j t - k)$:

$$\psi_k^j(t) = \begin{cases} \sqrt{2^j} & \frac{k}{2^j} \leq t < \frac{k}{2^j} + \frac{1}{2^{j+1}} \\ -\sqrt{2^j} & \frac{k}{2^j} + \frac{1}{2^{j+1}} \leq t < \frac{k+1}{2^j} \\ 0 & \text{otherwise} \end{cases}$$

Haar wavelet (3)

- $V^2 = \phi_0^2 \oplus \phi_1^2 \oplus \phi_2^2 \oplus \phi_3^2 = \phi_0^1 \oplus \phi_1^1 \oplus \psi_0^1 \oplus \psi_1^1$



- Easy to verify that (tutorial work):
  - $\left\langle \psi_k^j, \psi_{k'}^j \right\rangle = 0 \ k \neq k'$
  - $\left\langle \psi_k^j, \psi_k^{j'} \right\rangle = 0 \ j \neq j'$

Haar wavelet (4)

Transition from resolution $j+1$ to j (compression)

- ϕ_k^j scaling functions: approximation at resolution j
- ψ_k^j wavelet functions: details at resolution j
- By definition of ϕ_k^j and ψ_k^j, we have:

$$\phi_k^j = \frac{\phi_{2k}^{j+1} + \phi_{2k+1}^{j+1}}{\sqrt{2}} \quad \psi_k^j = \frac{\phi_{2k}^{j+1} - \phi_{2k+1}^{j+1}}{\sqrt{2}} \tag{6}$$

- And: $x^{j+1}(t) = \sum_{k=0}^{2^j-1} s_k^j \phi_k^j(t) + \sum_{k=0}^{2^j-1} d_k^j \psi_k^j(t) = \sum_{k=0}^{2^{j+1}-1} s_k^{j+1} \phi_k^{j+1}(t)$
- We derive:

$$s_k^j = \frac{s_{2k}^{j+1} + s_{2k+1}^{j+1}}{\sqrt{2}} \quad d_k^j = \frac{s_{2k}^{j+1} - s_{2k+1}^{j+1}}{\sqrt{2}}$$

Haar wavelet (5)

Transition from resolution j to $j+1$ (decompression)

- Inversion of system (6)

$$\phi_{2k}^{j+1} = \frac{\phi_k^j + \psi_k^j}{\sqrt{2}} \quad \phi_{2k+1}^{j+1} = \frac{\phi_k^j - \psi_k^j}{\sqrt{2}}$$

- We have: $x^{j+1}(t) = \sum_{k=0}^{2^j-1} s_k^j \phi_k^j(t) + \sum_{k=0}^{2^j-1} d_k^j \psi_k^j(t) = \sum_{k=0}^{2^{j+1}-1} s_k^{j+1} \phi_k^{j+1}(t)$
- We derive:

$$s_{2k}^{j+1} = \frac{s_k^j + d_k^j}{\sqrt{2}} \quad s_{2k+1}^{j+1} = \frac{s_k^j - d_k^j}{\sqrt{2}}$$

Content

The discrete wavelet transform (1)

- Haar: scaling and details functions or coefficients at a given resolution derive from a linear combination of scaling and wavelet functions or coefficients at the superior resolution.
  This can be generalized...
- $V^0 \subset V^1$:
  - then $\phi(t) \in V^0 \Rightarrow \phi(t) \in V^1$
  - then $\exists h(n)$ such as $\phi(t) = \sum_n h(n)\phi_n^1(t)$
  - then $\phi(t) = \sqrt{2}\sum_n h(n)\phi(2t - n)$
- This holds for any $V^{j-1} \subset V^j$ and generalizes as follow:
  - $\phi_k^{j-1}(t) \in V^{j-1} \Rightarrow \phi_k^{j-1}(t) \in V^j$
  - $\phi_k^{j-1}(t) = \sum_n h(n)\phi_{n+2k}^j(t) = \sqrt{2^j}\sum_n h(n)\phi(2^j t - n - 2k)$
- Consequence on approximation coefficients:
  - $s_k^{j-1} = \left\langle x, \phi_k^{j-1} \right\rangle$
  - $s_k^{j-1} = \sum_n h(n) \left\langle x, \phi_{n+2k}^j(t) \right\rangle$
  - $s_k^{j-1} = \sum_n h(n) s_{n+2k}^j = \sqrt{2}\sum_{n'} h(n' - 2k) s_{n'}^j$
  - $s_k^{j-1} = h^* \star s^j(2k)$ (with h^* the mirror filter of h)
- $\phi \leftrightarrow h$

The discrete wavelet transform (2)

- Same discussion on details subspaces W^j
- $W^0 \subset V^1$:
  - $\psi(t) \in W^0 \Rightarrow \psi(t) \in V^1$
  - $\exists g$ such as $\psi(t) = \sum_n g(n)\phi_n^1(t) = \sqrt{2}\sum_h g(n)\phi(2t-n)$
- Superior resolutions:
  - $\psi_k^{j-1}(t) = \sum_n g(k)\phi_{n+2k}^j(t) = \sqrt{2^j}\sum_n g(n)\phi(2^j t - n - 2k)$
- Consequence on details coefficients:
  - $d_k^{j-1} = \left\langle x, \psi_k^{j-1} \right\rangle$
  - $d_k^{j-1} = \sum_n g(n) \left\langle x, \phi_{n+2k}^j \right\rangle$
  - $d_k^{j-1} = \sum_n g(n) s_{n+2k}^j$
  - $d_k^{j-1} = g^* \star s^j(2k)$
- $\psi \leftrightarrow g$
- Reconstruction:

$$s_k^{j+1} = \sum_n s_n^j h(k-2n) + \sum_m d_m^j g(k-2m)$$

The discrete wavelet transform (3)

Link between ϕ and h

- Build an orthonormal basis, two ways: choose ϕ (see Haar scaling function), or choose h
- Indeed:
  - ϕ and h are linked ($V^0 \subset V^1$): $\phi(t) = \sqrt{2} \sum_n h(n)\phi(2t-n)$
  - Apply FT on previous equation, introduce $\omega = 2\pi f$, denote $\Phi = FT(\phi)$, and $H(\omega) = \sum_n h(n) e^{-in\omega}$
  - We have:

$$\Phi(\omega) = \frac{1}{\sqrt{2}} \Phi\left(\frac{\omega}{2}\right) H\left(\frac{\omega}{2}\right) = \prod_{j=1}^{+\infty} \frac{1}{\sqrt{2}} H\left(\frac{\omega}{2^j}\right)$$

- Then H can be derived from Φ and reciprocally
- H is a low-pass filter. Indeed:
  - $H(0) = \sqrt{2}\Phi(0)/\Phi(0/2) = \sqrt{2}$ ($\Phi(0) \neq 0$ because $\int \phi(t)dt$ can not be null)
  - from relation between Φ and H, it can been shown that $|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2$, then $H(\pi) = 0$

The discrete wavelet transform (4)

Link between ψ and g, and h!

- Similarly, we have ($W^0 \subset V^1$): $\psi(t) = \sqrt{2}\sum_n g(n)\phi(2t-n)$ then:

$$\Psi(\omega) = \frac{1}{\sqrt{2}}\Phi\left(\frac{\omega}{2}\right)G\left(\frac{\omega}{2}\right) = \prod_{j=1}^{+\infty}\frac{1}{\sqrt{2}}G\left(\frac{\omega}{2^j}\right)$$

- G is a high-pass filter:
  - $G(0) = 0$ as $\Psi(0) = \int \psi(t)dt = 0$ by definition (oscillating)
  - Again: $|G(\omega)|^2 + |G(\omega+\pi)|^2 = 2$ and then $G(\pi) = \sqrt{2}$
- Moreover, one can prove that:
  - $G(\omega) = -\Lambda(\omega)\bar{H}(\omega+\pi)$ with Λ verifying this two conditions: $\Lambda(\omega+2\pi) \pm \Lambda(\omega) = 0$
  - A solution is $\Lambda(\omega) = -e^{-i\omega}$
- Finally g can be derived from h:

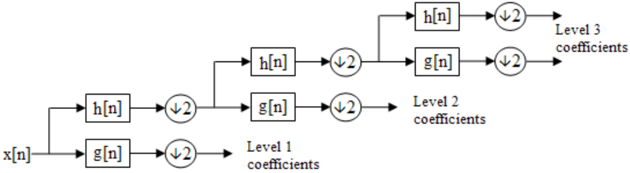
$$\begin{aligned} G(\omega) &= -e^{-i\omega}\bar{H}(\omega+\pi) \\ g(n) &= (-1)^n h(1-n) \end{aligned} \quad (7)$$

- g is the conjugate and mirror filter of h

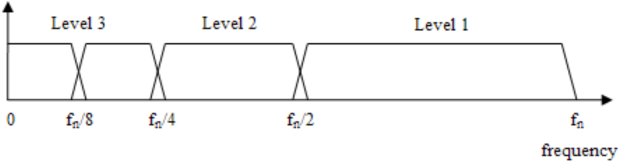
The discrete wavelet transform (5)

Cascade algorithm with mirror and conjugate filters

- The DWT is efficiently implemented using a series of low and high-pass filtering and sub-sampling (due to dyadic nature of MRA)



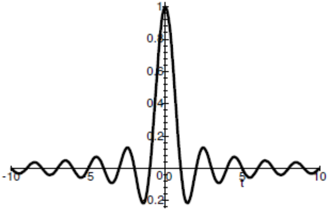
- low-pass filtering: low frequencies are captured with accurate frequency resolution, but poor time resolution
- high-pass filtering: high frequencies are captured with poor frequency resolution but an accurate time resolution

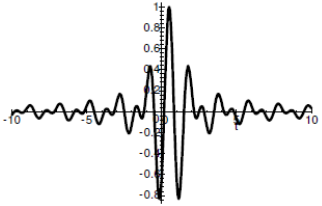


Other wavelet transforms (1)

Shannon wavelet

- We only know Haar wavelet: $h(n) = \begin{pmatrix} 1 & 1 \end{pmatrix}$, and $g(n) = \begin{pmatrix} 1 & -1 \end{pmatrix}$ (Important: do not forget to divide by $\sqrt{2}$ in practice!)
- Shannon wavelet (dual of Haar):
  - Haar: $\phi(t) = \text{Rect}(t) \Rightarrow \Phi(f) = \text{sinc}(\pi f)$
  - Shannon: $\phi(t) = \text{sinc}(\pi t) \Rightarrow \Phi(\omega) = \text{Rect}(\omega)$
  - We derive $H(\omega)$ then h: $h(n) = \text{sinc}\left(\frac{n\pi}{2}\right)$
  - then $G(\omega)$ from $g(n) = (-1)^n h(1-n) = (-1)^n \text{sinc}\left(\frac{(1-n)\pi}{2}\right)$
  - then $\Psi(\omega)$ and finally $\psi(t) = \frac{\cos(\pi t) - \sin(2\pi t)}{\pi t}$





Other wavelet transforms (2)

Daubechies wavelet (1)

- Motivation: build a basis with n null moments and compact support
- ψ has n null moments if:

$$\int_{\mathbb{R}} t^k \psi(t) dt = 0 \quad \forall k = 1, \cdots, n$$

- In other words: $\langle \psi(t), t^k \rangle = 0$, the mother wavelet is orthogonal to polynomials of degree $\leq n$
- Interest: the more a wavelet function has null moments, the more the signal representation is sparse. Essential property for compression.
- Properties of wavelet basis having many null moments:
  - the scaling function better approximates smooth signals
  - the wavelet function is dual: it better captures signal discontinuities

Other wavelet transforms (3)

Daubechies wavelet (2)

- Daubechies with 4 null moments (denoted D_4 or `db2` with Matlab)
- Filters h et g are of length 4
- If $h = (h_0, h_1, h_2, h_3)$ then $g = (h_3, -h_2, h_1, -h_0)$ (eq.(7))
- Constraints to determine the coefficients:
  - ψ of null mean $\Rightarrow h_3 - h_2 + h_1 - h_0 = 0$
  - ψ with 4 null moments $\Rightarrow h_3 - 2h_2 + 3h_1 - 4h_0 = 0$
  - $\langle \psi(t), \psi(t-1) \rangle = 0 \Rightarrow h_1 h_3 + h_2 h_0 = 0$
  - $\|\phi\| = 1 \Rightarrow h_0 + h_1 + h_2 + h_3 = 2$
- We find: $h_0 = \frac{1+\sqrt{3}}{4} \quad h_1 = \frac{3+\sqrt{3}}{4} \quad h_2 = \frac{3-\sqrt{3}}{4} \quad h_3 = \frac{1-\sqrt{3}}{4}$

Other wavelet transforms (3)

Daubechies wavelet (3)

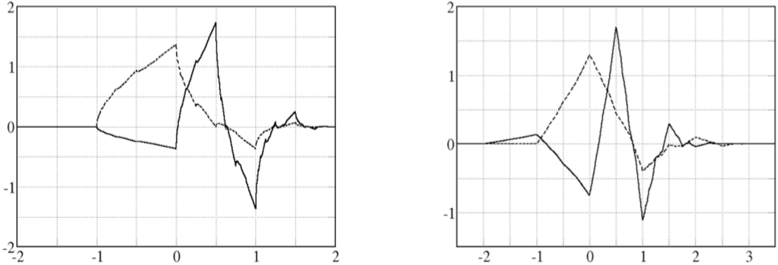


Figure: Daubechie scaling and wavelet functions with 4 null moments (db2) and 6 null moments (db3)

Content

2-D DWT for images

- 2-D Haar decomposition for a 2-D signal
- Two approaches:
  - the standard decomposition: 1-D DWT on one direction (lines), than 1-D DWT on the other direction (columns)
  - non standard decomposition: the 1-D DWT is alternated on lines and columns
  - both approaches lead to two specific 2-D Haar bases
- Advantages:
  - standard: only 1-D transforms
  - non standard, faster: $\frac{8}{3}(n^2 - 1)$ operations against $4(n^2 - n)$ for standard one

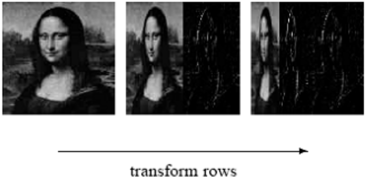
2-D DWT: standard decomposition (1)

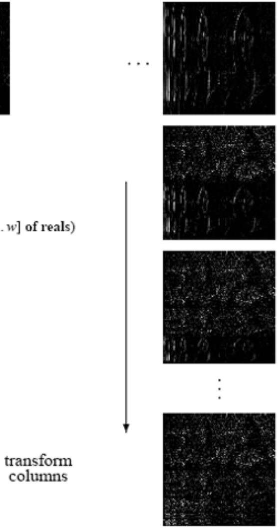
- Basis of the Haar standard decomposition is a tensor product between the 1-D bases:

$$\Psi_{k,k'}^{j,j'}(x,y) = \psi_k^j(x)\psi_{k'}^{j'}(y)$$

- Algorithm:
  1. apply a DWT on each line to obtain an intermediary image, repeat up to the finest resolution $j = 0$.
  2. then, apply a DWT on each column of this image, repeat up to the finest resolution
- we obtain an unique approximation coefficient and a set of details coefficients for all resolutions

2-D DWT: standard decomposition (2)






```
procedure StandardDecomposition(C: array [1..h, 1..w] of reals)
    for row ← 1 to h do
        Decomposition(C[row, 1..w])
    end for
    for col ← 1 to w do
        Decomposition(C[1..h, col])
    end for
end procedure
```

2-D DWT: standard decomposition (3)

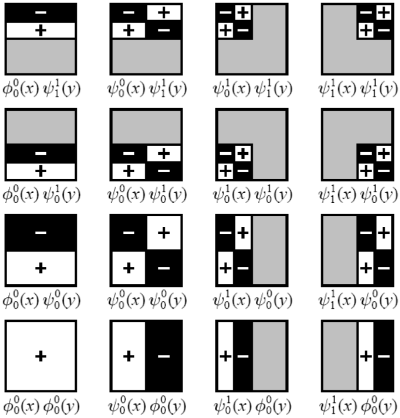


Figure: Haar standard basis

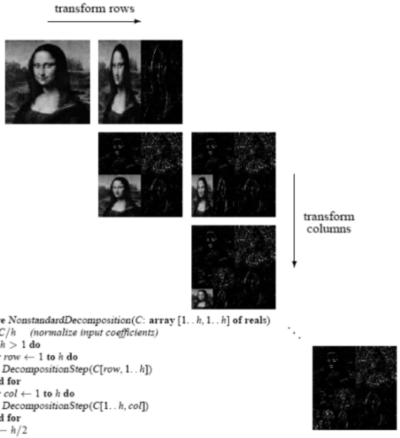
2-D DWT: non standard decomposition (1)

- Principle: perform an MRA of $L^2(\mathbb{R}^2)$
- Let's define $\mathcal{V}^j = V^j \otimes V^j$
- The details spaces are $\mathcal{W}^j$ such as $\mathcal{V}^{j+1} = \mathcal{V}^j \oplus \mathcal{W}^j$
- Then, we have:

$$\begin{aligned}
\mathcal{V}^{j+1} &= V^{j+1} \otimes V^{j+1} \\
&= (V^j \oplus W^j) \otimes (V^j \oplus W^j) \\
&= (V^j \otimes V^j) \oplus (W^j \otimes V^j) \oplus (V^j \otimes W^j) \oplus (W^j \otimes W^j) \\
&= \mathcal{V}^j \oplus \mathcal{W}^j
\end{aligned}$$

- Basis of $\mathcal{W}^j$: $\psi_k^j(x)\phi_{k'}^j(y), \phi_k^j(x)\psi_{k'}^j(y), \psi_k^j(x)\psi_{k'}^j(y), \quad k, k' \in \mathbb{Z}$

2-D DWT: non standard decomposition (2)




```
procedure NonstandardDecomposition(C: array [1..h, 1..h] of reals)
    C ← C/h    (normalize input coefficients)
    while h > 1 do
        for row ← 1 to h do
            DecompositionStep(C[row, 1..h])
        end for
        for col ← 1 to h do
            DecompositionStep(C[1..h, col])
        end for
        h ← h/2
    end while
end procedure
```

The DWT is alternated on lines and columns:

1. one iteration of 1-D DWT on each lines
2. one iteration of 1-D DWT on each column
3. repeat stages 1. and 2. on approximation image up to resolution $j = 0$

2-D DWT: non standard decomposition (3)

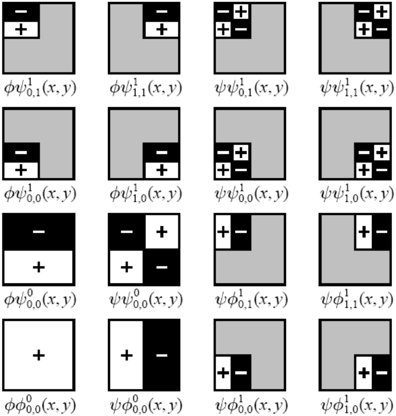
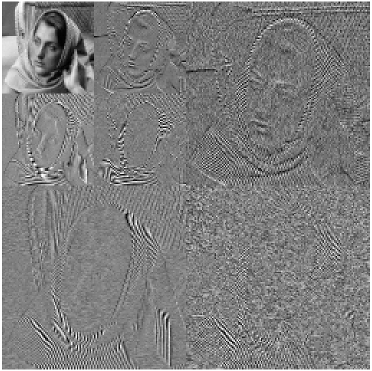


Figure: Base non standard de Haar 2-D

2-D DWT: Examples with Matlab[2]

```
[S1,H1,V1,D1] = dwt2(X,'haar');
imagesc([S1,H1;V1,D2])
```



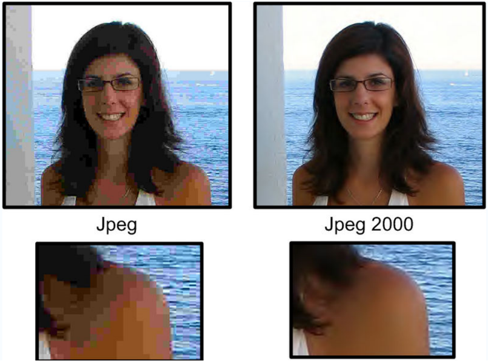
```
[S2,H2,V2,D2] = dwt2(S1,'haar');
imagesc([[S2,H2;V2,D2],H1;V1,D1])
```

[2] Python: use PyWavelets package

Content

Application: compression (1)

- Famous application (JPEG2000)
- JPEG compression (Fourier based): suppression of high frequencies $\Rightarrow$ edges are degraded (Gibbs phenomena)
- Suitable wavelet basis for edges representation: Haar (the Haar scaling function is basically an edge)

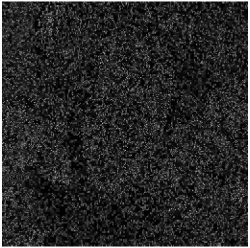


Application: compression (2)

- Principle: keep only the biggest details coefficients
- We apply an threshold:

Image

Reconstruction with
a threshold value of 10



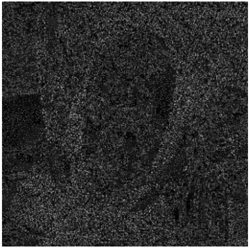
error: 1%

- 47% of details coefficients are zero (hence lesser than 10)
- without compression: 10% are zero

Application: compression (3)

Image

Reconstruction with
a threshold value of 40



error: 4.3 %

- 89% of the details coefficients are zero.
- Drawback (Haar): high compression rate makes appear blocs in the image

Application: denoising (1)

- Y image acquisition having an additive noise B
- Retrieve X such as

$$Y = X + B$$

- Practically, we look for an operator D minimizing the reconstruction error:

$$E(\|X - D(Y)\|) = \sum_{i=1}^{N} E(X(i) - D(Y)(i))^2 \tag{8}$$

- Many methods! Depending on the noise characteristics
- If B centered Gaussian, a wavelet filtering gives good results
- Method:
  - projection on a wavelet basis (encoding)
  - hard threshold: details coefficients lesser than threshold S are nullified
  - soft threshold: details coefficients lesser than threshold S are nullified, other are attenuated
  - How to choose S ?

Application: denoising (2)

- An optimal value minimizing (8) with respect to B be Gaussian of standard deviation σ:
$$S = \sigma\sqrt{2 \ln N}$$
- Estimation of σ?
$$\hat{\sigma} = \frac{M_s}{0,6745}$$
with M_s median value of details coefficients at the finest resolution
- Wavelet basis?
    - Haar
    - Daubechies
    - others: curvelets, ridgelets, ...

Application: denoising (3)

Image

Gaussian noise

Haar

Daubechies (db3)

Other applications

- 3-D mesh: approximation of a volume by decomposition on Haar wavelets
- Pattern recognition: for example, faces characterization, by projection on a wavelets basis
- Texture characterization and modeling
- Image watermarking: the trademark is projected on a wavelets basis, highest coefficients are retained and added to image details coefficients
- Sparse representation: wavelets allow sparse representations i.e. having a minimal number of coefficients