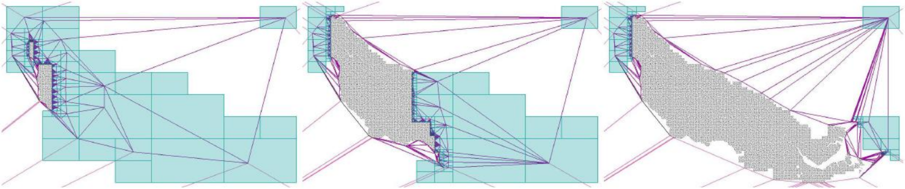

[ILSS06]

Delaunay complexes

Computational topology

Julien Tierny
jtierny@sci.utah.edu

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Well... where were we?

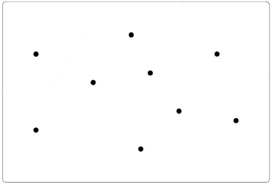
- Course Syllabus:
  - Basics on topological spaces;
  - **Simplicial complexes**;
  - Homology;
  - Topology abstractions (Reeb graph, MS-complex, etc.):
    - Computation algorithms;
    - Processing and simplification frameworks.

- Back to the past:
  - Complexes often come from real-life acquisitions;
  - Most of the time: point clouds;
  - How can we derive a valid simplicial complex out of that?
  - Next lectures:
    - **Delaunay complexes**;
    - Simulation of Simplicity;
    - Alpha shapes.

Outline

1. Basics:
   - Voronoï diagrams;
   - Delaunay triangulations;
   - Algorithm example in $\mathbb{R}^2$.
2. Generalization:
   - Power diagrams;
   - Regular triangulations;
   - Algorithm in arbitrary dimension [ES92].

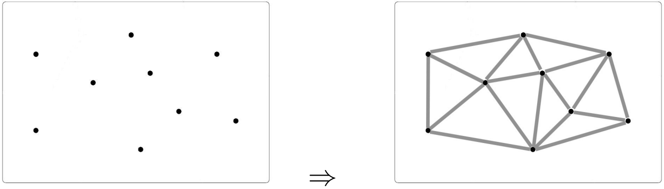
Problem formulation



$\Rightarrow$

- Input:
  - A set P of points in $\mathbb{R}^d$ in **general position**;
- Output:
  - A valid and unique d-**dimensional** simplicial complex $\mathcal{K}$;
  - Whose underlying space $|\mathcal{K}|$ is the **convex** hull of P:
    - The convex hull might not be a satisfactory approximation;
    - Can be formulated as a geometrical optimization problem;
    - Here, we *only* deal with combinatorial aspects.

Problem formulation



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Input description

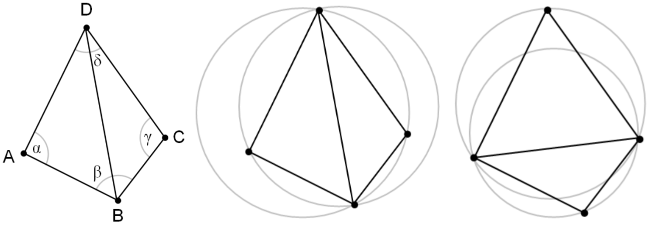
- Notion of general position:
  - P: set of points in $\mathbb{R}^d$;
  - The points of P are in general position if:
    - No $(d+1)$ points lie in a common $(d-1)$-dimensional plane;
    - Or no $(d+2)$ points lie in a common $(d-1)$-sphere.
  - Examples of forbidden configuration in $\mathbb{R}^2$:
    - Three co-linear points;
    - Four points on a same circle.
- Strong limitation, but still:
  - There's always a way to trick the data :)
  - Simulation of Simplicity [EM90]:
    - Slight perturbations on the data;
    - Transform forbidden configurations into non-degenerate ones;
    - Next class :)

Delaunay triangulations and mesh quality (intuition)

- A suitable property for surface mesh generation:
  - Having 2-simplices with regular geometry:
    - Equilateral triangles;
  - Enables to limit numerical errors when using the mesh:
    - Texture mapping;
    - Simulation, etc.

- What we can do *easily*:
  - Maximize the minimum angle of triangles.

Delaunay triangulations and angles (intuition)

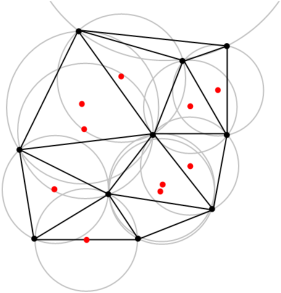


- Given a triangulation of 4 points in $\mathbb{R}^2$ (example):
  - Given the circumcircle $\mathcal{C}(ABD)$ of the triangle ABD;
  - A way to get rid of small angles in BCD:
    - Push C outside $\mathcal{C}(ABD)$.
  - We can only play on $\mathcal{K}$ (not on P), then, just guarantee that:
    - Given a 2-simplex $\sigma \in \mathcal{K}$, no point of P lie inside $\mathcal{C}(\sigma)$;
    - Just flip the edge BD into AC;
    - **Does it always make the trick?**

So what?

- According to this intuitive 2D example:
  - Given a set of points P in general position;
  - We need to compute a simplicial complex $\mathcal{K}$, such that:
    - P is the vertex set of $\mathcal{K}$;
    - Given a 2-simplex $\sigma \in \mathcal{K}$;
    - No point of P lie strictly inside of $\mathcal{C}(\sigma)$;
    - The dimension of $\mathcal{K}$ is 2;
    - The 2-simplices of $\mathcal{K}$ have at most 3 neighbors;

- Then:
  - We need to partition the space into cells:
    - Such that the vertices of those cells are the centers of the correct circumcircles;
    - The vertices of the cells have degree 3;
    - Notion of Voronoï diagram :)



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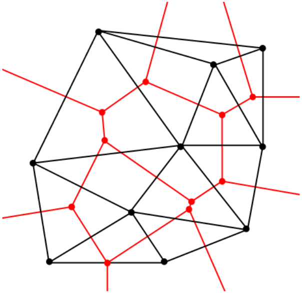
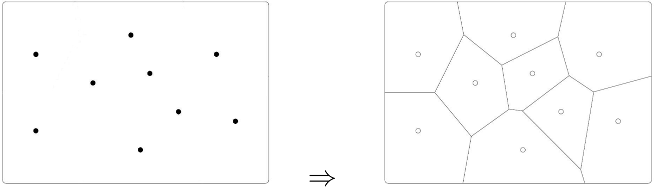


Voronoï diagrams

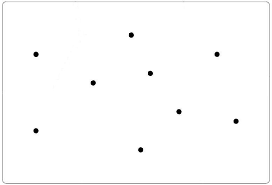


- Due to Georgy Voronoï (1907) but also met in Descartes's notes;
- Diversified applications (medecine, chemistry, climatology, etc.);

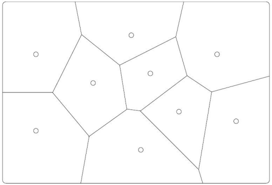
Voronoï diagrams (continued)

- Let $\pi_p(x)$ be :
  - The Euclidean distance between a point $x \in \mathbb{R}^d$ and a point $p \in P$;
- Chordale $\chi_{p,q}$ $(p, q \in P)$:
  - $\chi_{p,q}$: Locus of points $x \in \mathbb{R}^d$ with $\pi_p(x) = \pi_q(x)$;
  - $\chi_{p,q}$ is a $(d-1)$ plane;
- Half-spaces:
  - Let $H_{p,q}$ be the half space of points of $x \in \mathbb{R}^d$, such that:
    - $\pi_p(x) \leq \pi_q(x)$;
- The *Voronoï cell* $V(p)$ of $p \in P$ is:
  - $V(p) = \cap_{q \in P - \{p\}} H_{p,q}$;
  - or: $V(p) = \{x \in \mathbb{R}^d | \pi_p(x) \leq \pi_q(x), q \in P\}$.

Voronoï diagrams (continued)



$\Rightarrow$



- Properties:
  - $V(p)$ is a convex polyhedron in $\mathbb{R}^d$;
  - The intersection of the interiors of any two Voronoï cells is empty;
  - The union of all the Voronoï cells (Voronoï tessellation) covers $\mathbb{R}^d$;
  - **In 2D:**
    - **is it true that the vertices of the cells have always degree 3?**

Delaunay triangulation

- Given a set of points P in $\mathbb{R}^d$;
- The Delaunay triangulation $\mathcal{D}(P)$ of P is a triangulation of P;
- Such that:
  - There is no point of P in the inside of the circum-hypersphere of any d-simplex $\sigma \in \mathcal{D}(P)$;
- Let's use the Voronoï tessellation :)

Delaunay triangulation (continued)

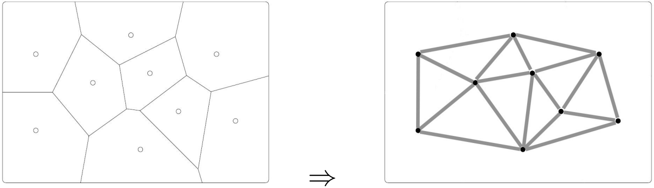
- Notion of *Nerve*:
  - Let $\mathcal{F}$ be a finite collection of sets.
  - The nerve $\mathcal{N}(\mathcal{F})$ of $\mathcal{F}$ consists of all subcollections whose sets have a non-empty common interesection:
    - $\mathcal{N}(\mathcal{F}) = \{X \subseteq \mathcal{F} | \cap X \neq \emptyset\}$;

Definition (Delaunay triangulation)

The Delaunay triangulation of a finite set of points P in $\mathbb{R}^d$ is isomorphic to the nerve of the collection of Voronoï cells:

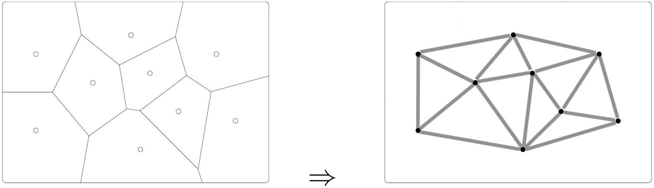
$$\mathcal{D}(P) = \{\sigma \subseteq P | \cap_{p \in \sigma} V(p) \neq \emptyset\}$$

In words



- The Delaunay triangulation can be seen as the dual of the Voronoï tessellation;
- It is composed of simplices σ:
  - That form the convex hull of sets of points of P,
  - whose Voronoï cells have non-empty intersections (adjacent cells);

Delaunay triangulations: properties



- Under the assumption of *general position* on P:
  - No $d+2$ points of P lie on a common $(d-1)$-sphere;
  - Then:
    - The center of these spheres are on the boundaries of the Voronoï cells;
    - No $d+2$ Voronoï cells have a non-empty common intersection;
    - (in 2D, degree-3 vertices);
    - Equivalently:
    - The dimension of any simplex of $\mathcal{D}(P)$ is at most d (see picture).
    - Valid d-dimensional simplicial complex!

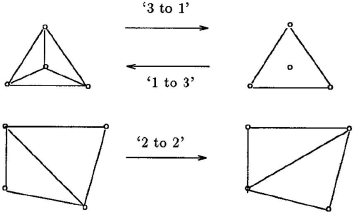
Algorithm example

- Incremental algorithm:

  1. Initial *artificial* simplex σ_0;
  2. Incremental insertion of a point $p \in P$:
     1. Identify the simplex containing p;
     2. Topological flip (locally guarantee Delaunay constraints);
     3. Related topological flips (globally guarantee Delaunay constraints);
     4. Records the flips in flip history;
  3. Remove the simplices having a vertex of the initial *artificial simplex* σ_0;

Notion of topological flip

- Aglorithm: incremental insertion plus Delaunay conditions;
- In 2D:
  - Insertion of a point in a simplex ('1 to 3');
  - Edge-flip: no point inside the circumsphere of a triangle ('2 to 2');
- In 3D:
  - Insertion of a point in a simplex ('1 to 4');
  - Triangle-flip: no point inside the circumsphere of a tet ('3 to 2');
- In dimension d: k d-simplices to $(d + 2 - k)$ d-simplices.



[ES92]

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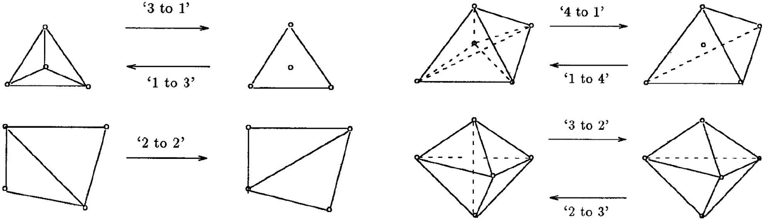
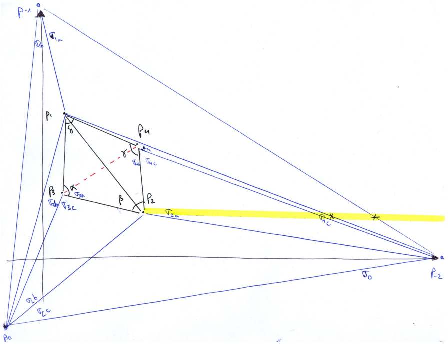
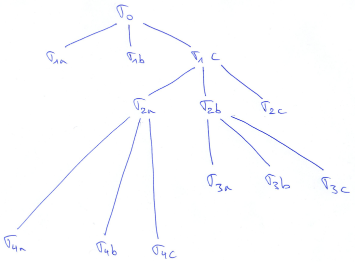


[ES92]

2D example

- Spatial hierarchy lookup (flip history);
- Point insertion;
- Topological flips (in 2D, edge opposite angles).





- Time complexity: $O(log(n))$ (look-up), repeated n times.

Generalizations

- Several ways to generalize Voronoï diagrams and Delaunay triangulations;
- Just play on π_p:
  - Non Euclidean metrics;
- In particular,
  - Point weighting (flexibility);
  - The power functions;
  - $\pi_p(x) = |xp|^2 - w_p$;
  - Direct application: wireless network design.
  - In general, point weighting allows for point importance characterization.

Chordales and half-spaces revisited

- Chordale $\chi_{p,q}$ $(p, q \in P)$:
  - Locus of points $x \in \mathbb{R}^d$ with $\pi_p(x) = \pi_q(x)$;
  - Then, $\chi_{p,q}$ is the following hyperplane:
    - $\chi_{p,q} = 2\sum_{i=1}^{d} x_i(q_i - p_i) + \sum_{i=1}^{d}(p_i^2 - q_i^2) - w_p + w_q = 0$;
- Half-spaces:
  - $H_{p,q}$: half-space of points $x \in \mathbb{R}^d$ with:
    - $\pi_p(x) \leq \pi_q(x)$;

Voronoï diagrams revisited: Power diagrams

- For each $p \in P$:
  - The *Power cell* $P(p)$ of $p \in P$ is:
    - $P(p) = \cap_{q \in P - \{p\}} H_{p,q}$
    - or $P(p) = \{x \in \mathbb{R}^d | \pi_p(x) \leq \pi_q(x), q \in P\}$.

- Properties:
  - $P(p)$ is a convex polygon;
  - The intersection of the interiors of any two power cells is empty;
  - The union of all the power cells covers $\mathbb{R}^d$;
  - The collection of power cells and their faces:
    - defines the cell complex $\mathcal{P}(P)$;
    - the *Power diagram* of P.

General position revisited

- Given the power functions $\pi_p, p \in P$;
- The context of general position slightly varies:

1. For every $d + 1$ weighted points in P:
   - There is a unique unweighted point $x \in \mathbb{R}^d, x \notin P$,
   - with the same power distance from all the $d + 1$ points.
2. For every $d + 2$ weighted points in P:
   - There is no such point;
   - (Generalization of the sphere condition).

Notion of orthogonality

- Two weighted points $p, z \in P$ are *orthogonal* if:
  - $|pz|^2 = w_p + w_z$;
  - Their Euclidean distance is equal to the sum of their power contribution;
  - Then:
    - $\pi_p(z) = w_z = -\pi_z(z)$;
    - and $\pi_z(p) = w_p = -\pi_p(p)$.
  - In other words, p and z are such that they do not influence each other.

- Let σ be a d-simplex of P:
  - Convex hull of $d + 1$ points of P;
  - There is a unique weighted point $z \in P$, such that:
    - z is orthogonal to all the weighted points of σ;
    - z is the *orthogonal center* of σ, noted $z(\sigma)$.

Global regularity

- $\pi_p(z) = w_z$ and $\pi_z(p) = w_p$, $\forall p \in \sigma$;
- σ is *globally regular* if:
  - $\pi_z(q) > w_q, \forall q \in P$;
  - Generalization of the property:
    - No point of P in the circumsphere of a d-simplex;
    - If all the weights of $p \in \sigma$ are zero,
    - The sphere centered in z with radius $\sqrt{w_z}$ is the circumsphere of σ.

Definition (Regular triangulations)

The regular d-simplices, together with their faces, define a simplicial complex called the *regular triangulation* of P, noted $\mathcal{R}(P)$.

- If all the weights of all points of P are zero, then:
  - $\mathcal{P}(P) = \mathcal{V}(P)$;
  - $\mathcal{R}(P) = \mathcal{D}(P)$.

Local regularity

- Let $\mathcal{T}$ be an arbitrary triangulation of P;
- Let σ' and σ'' be two adjacent d-simplices of $\mathcal{T}$:
  - $\sigma' \cap \sigma'' \neq \emptyset$;
  - $\sigma' \cap \sigma'' = \sigma$;
  - σ is a $(d-1)$-simplex.
  - Let $a \in P$, such that $a \in \sigma', a \notin \sigma''$;
  - Let $b \in P$, such that $b \in \sigma'', b \notin \sigma'$;
  - See picture (?)
  - Let $z' = z(\sigma')$, $\pi_{z'}(p) = w_z, \ \forall p \in \sigma'$;
  - σ is *locally regular* in $\mathcal{T}$ if:
    - $w_b < \pi_{z'}(b)$;
- If all the $(d-1)$-simplices of $\mathcal{T}$ are locally regular, then $\mathcal{T} = \mathcal{R}(P)$:
  - This allows for incremental algorithms :)
  - This also gives the topological flip condition.

Topological flippability

- Let $T = \sigma' \cup \sigma''$;
- σ is *flippable* in T if:
  - *conv*(T) is the underlaying space —T— of T.
- Consider the d $(d-2)$-simplices of σ:
  - Such a $(d-2)$-simplex is *convex* if:
    - There is an hyperplane containing it;
    - Such that σ' and σ'' both lie on the same side of the hyperplane.
  - Otherwise, the $(d-2)$-simplex is *reflex*.

- $|T| = conv(T)$ if and only if:
  - All reflex $(d-2)$-simplices of σ have degree 3;
  - Each is exactly incident to 3 $(d-1)$-simplices.

- Then:
  - The geometrical realization in $\mathbb{R}^d$ of $\mathcal{R}(P)$ is guaranteed;
  - This guarantees that $\mathcal{R}(P)$ is a d dimensional simplicial complex.

Incremental algorithm for Regular triangulations

1. Initial *artificial* d-simplex:
   - $\sigma_0 = conv(\{p_{-d}, \ldots, p_0\})$;
   - $p_{ij} = 0$ if $-i > j$;
   - $p_{ij} = +\infty$ if $-i = -j$;
   - $p_{ij} = -\infty$ if $-i < j$.
2. Incremental insertion:
   - Spatial lookup for the d-simplex σ_T containing p_i (flip history);
   - If $\mathcal{R}(T \cup \{p_i\}) \neq \sigma_T$ (locally non-regular):
     - Topological flip $T \cup \{p_i\}$;
     - While there remains locally non-regular $(d-1)$-simplices adjacent to p_i, flip them (stack).
3. Remove the simplices having a vertex in the initial *artificial* simplex.

- Same algorithm as in the 2D example;
- Time complexity: $O(nlog(n)) + n^{d/2}$.

Conclusion

- Given a point cloud P of $\mathbb{R}^d$:
  - We showed how to realize a d dimensional simplicial complex being a triangulation of P;
  - The underlaying space of this triangulation is the convex hull of P.

- We generalized it to weighted point clouds.

- Still!
  - This is only a combinatorial solution to shape reconstruction from point clouds;
  - Only the validity of the simplicial complex is guaranteed;
  - For example, reliable surface reconstruction from point clouds in $\mathbb{R}^3$ is still an active geometry research topic!

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