Appendix A. 2D ARAP MLS transformation gradient

We provide an analytic expression of the gradient of as-rigid-as-possible planar maps, for a given constraint \( q \). A closed-form formula for planar similarities is provided in [1]. For some given constraints \( p_i \) and their mapping \( q_i = \hat{\psi}_S(p_i) \), for each \( x \in \mathcal{D} \) with \((u, v)\) coordinate, the similarity transformation, noted \( \hat{\psi}_S \), is given by:

\[
\hat{\psi}_S(x) = q_* + \frac{\sum_i w_i \hat{q}_i \left( \frac{\hat{p}_i}{-\hat{p}_i} \right) (x - p_*)^T}{\mu_S}
\]

where \( w_i = \frac{1}{(p_i - x)^2} \), \( p_* = \sum_i w_i p_i \), \( q_* = \sum_i w_i q_i \), \( \hat{p}_i = p_i - p_* \) and \( \hat{q}_i = q_i - q_* \). \( \mu_S \) denotes the transpose and \((u, v)^T = (-v, u)\) and \( \mu_S = \sum_i w_i \hat{p}_i \hat{p}_i^T \).

A theorem is provided in [1], noticing that if locally the similarity can be re-written as a rotation matrix, it minimizes the energy functional of rigid transformations (see theorem 2.1). Then, the closed-form formula for planar ARAP transformations \( \hat{\psi}_R \) is the same as Eq. A.1, except that \( \mu_S \) is switched for \( \mu_R \):

\[
\mu_R = \sqrt{\left( \sum_i w_i \hat{q}_i \hat{p}_i^T \right)^2 + \left( \sum_i w_i \hat{q}_i \hat{p}_i^T \right)^2}
\]

For clarity, we derive the first component \((u)\) of \( \hat{\psi}_R \), wrt \( u \) \( (i) \), then wrt \( v \) \( (ii) \). The derivation of the second component (noted \( v \) of the gradient is obtained similarly and will be omitted. \( \hat{\psi}_R \) can be re-written in the form:

\[
\hat{\psi}_R(x) = q_* + \frac{\sum_i q_i A_i}{\mu_R}
\]

where \( A_i = w_i \left( \begin{array}{c} \hat{p}_i \\ -\hat{p}_i^\perp \end{array} \right) \left( \begin{array}{c} x - p_* \\ -(x - p_*)^\perp \end{array} \right)^T \) = \( \begin{array}{cc} a_i & 0 \\ 0 & a_i \end{array} \)

is an expression independent of \( q_i \) which can be pre-computed. Then, the \( u \)-component of \( \nabla_u \left( \hat{\psi}_R(x) \right) \), noted \( \nabla_u^k \left( \hat{\psi}_R(x) \right) \) is given by:

\[
\nabla_u^k \left( \hat{\psi}_R(x) \right) = \nabla_u^k q_{us} + \mu_R \nabla_u^k \left( \sum_i \hat{q}_i A_i \right)_u - (\sum_i \hat{q}_i A_i)_u \nabla_u^k \mu_R
\]

Notice \( \mu_R \) is a scalar and \((.)_u \) denotes the \( u \)-component of a vector. The derivation of \( q_{us} \) is straightforward (an infinitely small variation in \( q_{us} \) will be multiplied by \( \frac{w_k}{\sum_i w_i} \) while the \( v \) component remains unchanged):

\[
\nabla_u^k q_{us} = \left( \frac{w_k}{\sum_i w_i} 0 \right)
\]

\[
\left( \sum_i \hat{q}_i A_i \right)_u = \left( \sum_i \hat{q}_i u \right) a_i^0 + \left( \hat{q}_i v \right) a_i^1
\]

\[
\nabla_u^k \left( \sum_i \hat{q}_i A_i \right)_u = a_i^0 - w_k \sum_i (\sum_i a_i^0 / w_i)
\]

If we re-write \( \mu_R \) as \( \mu_R = \sqrt{(\mu_0^2 + \mu_1^2)} \), then we have:

\[
\nabla_u^k \mu_R = \frac{1}{2} \left( \mu_0^2 + \mu_1^2 \right)^{-\frac{1}{2}} (2\mu_0 \nabla_u^k \mu_0 + 2\mu_1 \nabla_u^k \mu_1)
\]

\[
\mu_0 = \sum_i ((\hat{q}_i)_u (\hat{p}_i)_u + (\hat{q}_i)_v (\hat{p}_i)_v)
\]

\[
\mu_1 = \sum_i ((\hat{q}_i)_u (\hat{p}_i)_u + (\hat{q}_i)_v (\hat{p}_i)_v)
\]

\[
\nabla_u^k \mu_0 = w_k (\hat{p}_k)_u - w_k \sum_i \frac{w_i (\hat{p}_i)_u}{\sum_i w_i}
\]

\[
\nabla_u^k \mu_1 = w_k (\hat{p}_k)_u - w_k \sum_i \frac{w_i (\hat{p}_i)_u}{\sum_i w_i}
\]
Thus (i), \( \nabla^k_u \left( \hat{\psi}_R(x) \right) \) (Eq. A.4) can be computed by combining the equations A.5, A.7, A.11 and A.12. Now, we need to derive again the \( u \) component of \( \hat{\psi}_R(x) \), but wrt \( v \):

\[
\begin{align*}
\nabla^k_v \left( \hat{\psi}_R(x) \right)_u &= \nabla^k_v q_u \\
+ \mu_R \nabla^k_v \left( \sum_i \hat{q}_i A_i \right)_u - \left( \sum_i \hat{q}_i A_i \right)_u \nabla^k_v \mu_R \end{align*}
\] (A.13)

\[
\begin{align*}
\nabla^k_v \left( \sum_i \hat{q}_i A_i \right)_u &= a^k_1 - w_k \sum_i \frac{a^i_4}{\sum_i w_i} \\
\nabla^k_v \mu_0 &= w_k (\hat{p}_k)_v - w_k \sum_i \frac{w_i (\hat{p}_i)_v}{\sum_i w_i} \\
\nabla^k_v \mu_1 &= w_k (\hat{p}_k^\perp)_v - w_k \sum_i \frac{w_i (\hat{p}_i^\perp)_v}{\sum_i w_i} \\
\end{align*}
\] (A.14)

(A.15)

(A.16)

Then (ii), \( \nabla^k_v \left( \hat{\psi}_R(x) \right)_u \) can be computed by combining the equations A.5, A.14, A.15 and A.16 into the equation A.13. Finally, to complete the gradient computation, the \( v \) component of \( \hat{\psi}_R \) needs to be derived twice in the same manner: (i) with a small variation of \( q_k \) in \( u \), (ii) with a small variation of \( q_k \) in \( v \). We do not detail this derivation since it is highly similar to the derivations detailed above (a noticeable difference is that \( a^0_i \) and \( a^1_i \) need respectively to be switched for \( a^2_i \) and \( a^3_i \), while the derivations of \( \mu_0 \) and \( \mu_1 \) do not change).

The computation of \( \nabla \hat{\psi}_R \) is very efficient. The following terms are computed offline (before the optimization), as soon as the \( p_i \) set is known: \( w_i \), \( \sum w_i \), \( A_i \), \( \hat{p}_i \) and \( \hat{p}_i^\perp \). The following terms are computed during the online reconstruction of \( \hat{\psi}_R \): \( q_i \), \( \mu_R \), \( \mu_0 \) and \( \mu_1 \). Thus, for a given constraint \( q_k \), the gradient computation algorithm only needs to directly evaluate the right hand side of equations A.5, A.7, A.11, A.12, A.14, A.15 A.16, and A.8 (plus the same operations for the derivation of the \( v \) component of \( \hat{\psi}_R \)). In practice, this is achieved in twice the time necessary to reconstruct the map online.

References