A Note on Structured Pseudospectra

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Abstract

In this note, we study the notion of structured pseudospectra. We prove that for Toeplitz, circulant and symmetric structures, the structured pseudospectrum equals the unstructured pseudospectrum. We show that this is false for Hermitian and skew-Hermitian structures. We generalize the result to pseudospectra of matrix polynomials. Indeed, we prove that the structured pseudospectrum equals the unstructured pseudospectrum for matrix polynomials with Toeplitz, circulant, Hankel and symmetric structures. We conclude by giving a formula for structured pseudospectra of real matrix polynomials. The particular type of perturbations uses for this pseudospectra arises in control theory.

Key words: structured perturbation, pseudospectrum, polynomial matrix, Toeplitz matrix, circulant matrix, Hankel matrix, symmetric matrix

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1 Introduction and notation

The $\varepsilon$-pseudospectrum of a matrix $A$, denoted by $\Lambda_\varepsilon(A)$, is the subset of complex numbers consisting of all eigenvalues of all complex matrices within a distance $\varepsilon$ of $A$ (see [12, 13]). If the matrix $A$ has a certain structure (for example Toeplitz), it is natural to allow only perturbed matrices with the same structure. In this case, the structured $\varepsilon$-pseudospectrum of a structured
Toeplitz matrices \((t_{i-j})_{i,j=0}^{n-1}\)  

\[
\begin{pmatrix}
 t_0 & t_{-1} & \cdots & t_{1-n} \\
 t_1 & t_0 & \ddots & \vdots \\
 \vdots & \ddots & \ddots & t_{-1} \\
 t_{n-1} & \cdots & t_1 & t_0 \\
\end{pmatrix}
\]  

Hankel matrices \((h_{i,j})_{i,j=0}^{n-1}\)  

\[
\begin{pmatrix}
 h_0 & h_1 & \cdots & h_{n-1} \\
 h_1 & h_2 & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \vdots \\
 h_{n-1} & h_n & \cdots & h_{2n-2} \\
\end{pmatrix}
\]  

Circulant matrices \((v_i)_{i=0}^{n-1}\)  

\[
\begin{pmatrix}
 v_0 & v_{n-1} & \cdots & v_1 \\
 v_1 & v_0 & \ddots & \vdots \\
 \vdots & \ddots & \ddots & v_{n-1} \\
 v_{n-1} & \cdots & v_1 & v_0 \\
\end{pmatrix}
\]

Table 1: Three classes of structured matrices

<table>
<thead>
<tr>
<th>Structure</th>
<th>general</th>
<th>Toep</th>
<th>circ</th>
<th>Hankel</th>
<th>sym</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k)</td>
<td>(n^2)</td>
<td>(2n-1)</td>
<td>(n)</td>
<td>(2n-1)</td>
<td>((n^2 + n)/2)</td>
</tr>
</tbody>
</table>

Table 2: Number of independent parameters

matrix \(A\), denoted by \(\Lambda^\text{struct}_\varepsilon(A)\), is the subset of complex numbers consisting of all eigenvalues of all complex structured matrices within a distance \(\varepsilon\) of \(A\).

In this paper, we are mainly concerned with the following linear structures corresponding to the set of Toeplitz, circulant, Hankel and symmetric matrices, see Table 1. One will find in Table 2 the number of independent parameters for structures we consider in this paper.

Throughout the paper, we denote by \(M_n(C)\) the set of complex \(n \times n\) matrices and \(M^\text{struct}_n(C)\) the set of structured complex matrices, struct as in (1.1). We endow these spaces with the 2-norm (also called the spectral norm) denoted by \(\| \cdot \|\).

Let us consider a matrix \(A \in M_n(C)\). We denote its spectrum by \(\Lambda(A)\). For a real \(\varepsilon > 0\), the \(\varepsilon\)-pseudospectrum of a matrix \(A \in M_n(C)\) is the set

\[
\Lambda_\varepsilon(A) = \{z \in C : z \in \Lambda(X) \text{ where } X \in M_n(C) \text{ and } \|X - A\| \leq \varepsilon\}.
\]

In the case where the matrix \(A\) has a certain structure, the entries are assumed to be defined according to this structure. This means that only per-
turbations on the entries are possible. For example, for a Toeplitz matrix, only $2n - 1$ coefficients are necessary to know the matrix and so only this $2n - 1$ coefficients may be perturbed. This justifies the introduction of structured pseudospectra. Given a matrix $A \in M_n^{\text{struct}}(\mathbb{C})$ with struct as in (1.1), the structured $\varepsilon$-pseudospectrum of $A$ is defined by

$$\Lambda_{\varepsilon}^{\text{struct}}(A) = \{ z \in \mathbb{C} : z \in \Lambda(X) \text{ where } X \in M_n^{\text{struct}}(\mathbb{C}) \text{ and } \| X - A \| \leq \varepsilon \}.$$ 

For $A \in M_n^{\text{struct}}(\mathbb{C})$ it is clear that we always have

$$\Lambda_{\varepsilon}^{\text{struct}}(A) \subseteq \Lambda_{\varepsilon}(A).$$

We are interested in the structures for which there is equality.

Up to our knowledge, structured pseudospectra (also called “spectral value sets”) have been first defined and studied with perturbations of the form

$$A \rightsquigarrow A + \Delta A = A + D\Theta E, \quad \Theta \in M_{l,q}(\mathbb{C}),$$

where $D \in M_{n,l}(\mathbb{C})$, $E \in M_{q,n}(\mathbb{C})$ are fixed matrices defining the structure of the perturbation (see [5, 11, 1]). The definition of structured pseudospectra we use in this note was first introduced by Böttcher, Grudsky and Kozak [3] for Toeplitz structure. They called it “Toeplitz” $\varepsilon$-pseudospectrum in [3] and Toeplitz-structured pseudospectrum in [2]. In [3], they considered banded Toeplitz matrices only and hence restricted themselves to defining $\Lambda_{\varepsilon}^{\text{Toep}[r,s]}(A)$ for $A \in M_n^{\text{Toep}[r,s]}(\mathbb{C})$ where Toep[$r, s$] stands for Toeplitz matrices with at most $r$ nonzero superdiagonals and at most $s$ nonzero subdiagonals. They established that $\Lambda_{\varepsilon}(A)$ may be different than $\Lambda_{\varepsilon}^{\text{Toep}[r,s]}(A)$. In this note, we show equality for $r = s = n$. Moreover, we extend the definition to other structures like circulant and symmetric.

The paper is organized as follows. In Section 2, we recall results on structured distance to singularity. In Section 3, we prove that for struct $\in \{\text{Toep, circ, sym}\}$, the structured pseudospectrum equals the unstructured pseudospectrum. Then, we study the case of structures Hermitian and skew-Hermitian. We prove that the equality of the structured and unstructured pseudospectrum does not hold for these structures. In Section 4, we generalize the previous results to pseudospectra of matrix polynomials with struct $\in \{\text{Toep, circ, Hankel, sym}\}$. We also consider structured pseudospectra of real matrix polynomials.
2 Results on structured distance to singularity

In this section, we recall some results on structured distance to singularity. Given a nonsingular matrix \( A \in M_n(\mathbb{C}) \), we define the distance to singularity by

\[
d(A) = \min\{\|\Delta A\| : A + \Delta A \text{ singular}, \Delta A \in M_n(\mathbb{C})\}.
\]

(2.2)

For a nonsingular matrix \( A \in M_n^{\text{struct}}(\mathbb{C}) \), we define the structured distance to singularity by

\[
d^{\text{struct}}(A) = \min\{\|\Delta A\| : A + \Delta A \text{ singular}, \Delta A \in M_n^{\text{struct}}(\mathbb{C})\}.
\]

(2.3)

Rump has proved in [9, Thm 12.2] that the two distances \( d(A) \) and \( d^{\text{struct}}(A) \) are equal for \( \text{struct} \in \{\text{Toep}, \text{circ}, \text{Hankel}\} \).

Theorem 2.1 (Rump [9, Thm 12.2]). Let nonsingular \( A \in M_n^{\text{struct}}(\mathbb{C}) \) be given for \( \text{struct} \in \{\text{Toep, circ, Hankel}\} \). Then we have

\[
d(A) = d^{\text{struct}}(A) = \|A^{-1}\|^{-1} = \sigma_{\text{min}}(A).
\]

Here, \( \sigma_{\text{min}}(A) \) denotes the smallest singular value of \( A \) and it is well-known that \( \sigma_{\text{min}}(A) = \|A^{-1}\|^{-1} \). The same property occurs for the symmetric structure. Before stating the result, we will need the two following lemmas.

Lemma 2.2 (Takagi’s factorization). If \( A \) is complex symmetric \((A^T = A)\), then there exist a unitary matrix \( U \) and a real nonnegative diagonal matrix \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n) \) such that \( A = U\Sigma U^T \).

We refer to [7, Cor. 4.4.4] for a proof.

Lemma 2.3 (Rump [9, Lem. 10.1]). Let \( x \in \mathbb{C}^n \) be given. Then there exists \( A \) complex symmetric such that \( Ax = \overline{x} \) and \( \|A\| = 1 \).

The next result can be found in Tisseur and Graillat [10]. For completeness, we recall the proof.

Theorem 2.4 (Tisseur and Graillat [10]). Let nonsingular \( A \in M_n^{\text{struct}}(\mathbb{C}) \) be given for \( \text{struct} \) being symmetric. Then we have

\[
d(A) = d^{\text{struct}}(A) = \|A^{-1}\|^{-1} = \sigma_{\text{min}}(A).
\]
Proof. Obviously, we have $d_{\text{struct}}(A) \geq d(A) = \|A^{-1}\|^{-1} = \sigma_{\min}(A)$, and then it remains to show that $(A + \Delta A)x = 0$ for some $x \neq 0$ and $\Delta A$ symmetric with $\|\Delta A\| = \sigma_{\min}(A)$. Let $A = U\Sigma U^T$ be the Takagi’s factorization of $A$ where $U$ is unitary and $\Sigma$ is diagonal with nonnegative entries (see Lemma 2.2). Let $x$ be the column of $U$ corresponding to the smallest entry in $\Sigma$. Then $A\overline{x} = \sigma_{\min}(A)x$. By Lemma 2.3 there exists a symmetric matrix $C$ such that $C\overline{x} = x$ and $\|C\| = 1$. Let $\Delta A = -\sigma_{\min}(A)C$. Then $\Delta A$ is symmetric, $\|\Delta A\| = \sigma_{\min}(A)$ and

$$(A + \Delta A)\overline{x} = \sigma_{\min}(A)x - \sigma_{\min}(A)x = 0$$

so that $A + \Delta A$ is singular. \hfill \Box

3 Structured pseudospectrum equals unstructured pseudospectrum

The following lemma shows that the $\varepsilon$-pseudospectrum is linked to the distance to singularity. This is a well-known result.

Lemma 3.1. Given $\varepsilon > 0$ and $A \in M_n(\mathbb{C})$, the $\varepsilon$-pseudospectrum satisfies

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : d(A - zI) \leq \varepsilon\}.$$

In this section, we deal with

$$\text{struct} \in \{\text{Toep}, \text{circ}, \text{sym}\} \quad (3.4)$$

As we have seen before, we have $d(A) = d_{\text{struct}}(A)$ for $A \in M_n^{\text{struct}}(\mathbb{C})$. Hence, it is sufficient to prove that

$$\Lambda_\varepsilon^{\text{struct}}(A) = \{z \in \mathbb{C} : d_{\text{struct}}(A - zI) \leq \varepsilon\},$$

in order to conclude that $\Lambda_\varepsilon(A) = \Lambda_\varepsilon^{\text{struct}}(A)$ for a given matrix $A \in M_n^{\text{struct}}(\mathbb{C})$. This is the aim of the following lemma.

Lemma 3.2. Given $\varepsilon > 0$ and $A \in M_n^{\text{struct}}(\mathbb{C})$ with struct in (3.4), the structured $\varepsilon$-pseudospectrum satisfies

$$\Lambda_\varepsilon^{\text{struct}}(A) = \{z \in \mathbb{C} : d_{\text{struct}}(A - zI) \leq \varepsilon\}.$$

The proof is very similar to the one of Lemma 3.1 but we have to pay attention to keep the structure.
Proof. Let \( z \in \Lambda^\text{struct}_\varepsilon(A) \). This means that there exists \( X \in M_n^\text{struct}(\mathbb{C}) \) such that \( z \in \Lambda(X) \) and \( \|X - A\| \leq \varepsilon \). Hence the matrix \( X - zI \) is singular and

\[
\| (X - zI) - (A - zI) \| = \|X - A\| \leq \varepsilon.
\]

Moreover, since for our structure \( zI \in M_n^\text{struct}(\mathbb{C}) \) for \( z \in \mathbb{C} \), we have \( X - zI \in M_n^\text{struct}(\mathbb{C}) \). From the definition of the distance to singularity we obtain that

\[
d^\text{struct}(A - zI) \leq \varepsilon.
\]

Conversely, let \( z \) be such that \( d^\text{struct}(A - zI) \leq \varepsilon \). Then there exists \( X \in M_n^\text{struct}(\mathbb{C}) \) such that \( X \) is singular and \( \|(A - zI) - X\| \leq \varepsilon \). Let \( Y = X + zI \).

It follows that \( z \) is an eigenvalue of \( Y \) since \( Y - zI = X \) is singular. Moreover,

\[
\|Y - A\| = \|(A - zI) - X\| \leq \varepsilon
\]

and \( Y \in M_n^\text{struct}(\mathbb{C}) \) since \( zI \in M_n^\text{struct}(\mathbb{C}) \) for \( z \in \mathbb{C} \). It follows that \( z \in \Lambda(Y) \) with \( \|Y - A\| \leq \varepsilon \), that is to say \( z \in \Lambda^\text{struct}_\varepsilon(A) \).

From Lemma 3.1 and 3.2 and Theorem 2.1 we deduce the following theorem.

**Theorem 3.3.** Given \( \varepsilon > 0 \) and \( A \in M_n^\text{struct}(\mathbb{C}) \) with \( \text{struct} \in \{\text{Toep, circ, sym}\} \), the \( \varepsilon \)-pseudospectrum and the structured \( \varepsilon \)-pseudospectrum satisfy

\[
\Lambda^\text{struct}_\varepsilon(A) = \Lambda_\varepsilon(A).
\]

Theorem 2.1 has also been established for Hermitian, skew-Hermitian and Hankel structures. Nevertheless the proof given previously does not hold for these structures since the scalar matrices \( zI \) for \( z \in \mathbb{C} \) do not have these structures.

In fact, we do not have equality between the structured and the unstructured pseudospectrum for the Hermitian and skew-Hermitian structures.

Let us begin with the Hermitian structure. Let \( A \in M_n(\mathbb{C}) \) be an Hermitian matrix. It is well-known that all its eigenvalues are real (see for example [7, p.104]). For all \( \Delta A \in M_n(\mathbb{C}) \), we have \( A + \Delta A \in M_n(\mathbb{C}) \).

As a consequence, it follows that

\[
\Lambda^\text{herm}_\varepsilon(A) \subseteq \mathbb{R}.
\]

This is not the case for \( \Lambda_\varepsilon(A) \). Let us show it on a simple example in \( M_2(\mathbb{C}) \).

Let us define

\[
A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \in M_2^\text{herm}(\mathbb{C}).
\]
If we choose $\varepsilon = 0.01$ and
\[
\Delta A = \begin{pmatrix} 0.001i & 0 \\ 0 & 0.001i \end{pmatrix} \in M_2(\mathbb{C}),
\]
then we have $\|\Delta A\| = 0.001$ and the eigenvalues of $A + \Delta A$ are $2.000000001 + 0.001i$ and $0.001i$. We deduce that $\Lambda_\varepsilon(A) \not\subseteq \mathbb{R}$.

The same phenomenon still appears with skew-Hermitian structure. Indeed, if $A \in M_n(\mathbb{C})$ is a skew-Hermitian matrix then its eigenvalues are purely imaginary numbers. For all $\Delta A \in M_n^{\text{skewherm}}(\mathbb{C})$, we have $A + \Delta A \in M_n^{\text{skewherm}}(\mathbb{C})$. As a consequence, it follows that
\[
\Lambda_\varepsilon^{\text{skewherm}}(A) \not\subseteq \mathbb{R}.
\]
This is not the case for $\Lambda_\varepsilon(A)$. Let us define
\[
A = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \in M_2^{\text{skewherm}}(\mathbb{C}).
\]
If we choose $\varepsilon = 0.01$ and
\[
\Delta A = \begin{pmatrix} 0.001 & 0 \\ 0 & 0.001 \end{pmatrix} \in M_2(\mathbb{C}),
\]
then we have $\|\Delta A\| = 0.001$ and the eigenvalues of $A + \Delta A$ are $0.001 + 2.000000001i$ and $0.001$. We deduce that $\Lambda_\varepsilon(A) \not\subseteq \mathbb{R}$.

4 Structured pseudospectra of matrix polynomials

This section deals with pseudospectra of matrix polynomials. We prove a result analogous to Theorem 3.3 for the pseudospectra of matrix polynomials in the first subsection. The second subsection is concerned with structured pseudospectra of real matrix polynomials taking into account only real perturbations.

4.1 Structured pseudospectra of complex matrices

The polynomial eigenvalue problem is to find the solutions $(x, \lambda) \in \mathbb{C}^n \times \mathbb{C}$ of
\[
P(\lambda)x = 0,
\]
where
\[ P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0, \]
with \( A_k \in M_n(C), k = 0 : m \). If \( x \neq 0 \) then \( \lambda \) is called an eigenvalue and \( x \) the corresponding eigenvector. The set of eigenvalues of \( P \) is denoted \( \Lambda(P) \).

We assume that \( P \) has only finite eigenvalues (and pseudoeigenvalues). Let us define
\[ \Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0, \]
where \( \Delta A_k \in M_n(C) \), \( k = 0 : m \).

If \( x \neq 0 \) then \( \lambda \) is called an eigenvalue and \( x \) the corresponding eigenvector. The set of eigenvalues of \( P \) is denoted \( \Lambda(P) \).

We define \( \Lambda_\varepsilon(P) = \{ \lambda \in C : (P(\lambda) + \Delta P(\lambda))x = 0 \text{ for some } x \neq 0 \} \) with \( \|\Delta A_k\| \leq \alpha_k \varepsilon, k = 0 : m \} \).

The nonnegative parameters \( \alpha_1, \ldots, \alpha_m \) allow freedom in how perturbations are measured. The following lemma is a reformulation of Lemma 2.1 in [11].

**Lemma 4.1.**
\[ \Lambda_\varepsilon(P) = \{ \lambda \in C : d(P(\lambda)) \leq \varepsilon p(|\lambda|) \}, \]
where \( p(x) = \sum_{k=0}^m \alpha_k x^k \).

**Proof.** Let \( \lambda \) in \( \Lambda_\varepsilon(P) \). It means that there exists \( \Delta P(\lambda) \in M_n(C) \) such that \( \|\Delta A_k\| \leq \alpha_k \varepsilon, k = 0 : m \) and \( P(\lambda) + \Delta P(\lambda) \) is singular. It follows from the definition of the distance \( d \) that \( d(P(\lambda)) \leq \|\Delta P(\lambda)\| \). Since
\[ \|\Delta P(\lambda)\| \leq \sum_{k=0}^m |\lambda|^k \alpha_k \varepsilon = \varepsilon p(|\lambda|), \]
we have \( d(P(\lambda)) \leq \varepsilon p(|\lambda|) \).

Conversely, let \( \lambda \in C \) be such that \( d(P(\lambda)) \leq \varepsilon p(|\lambda|) \). It means that there exists \( X \in M_n(C) \) such that \( \|X\| \leq \varepsilon p(|\lambda|) \) and \( P(\lambda) + X \) singular. Let us define \( \Delta A_k \) by
\[ \Delta A_k = \text{sign}(\lambda^k) \alpha_k p(|\lambda|)^{-1} X, \]
where for complex \( z \) we define
\[ \text{sign}(z) = \begin{cases} \overline{z}/|z|, & z \neq 0, \\ 0, & z = 0. \end{cases} \]

Then
\[ \Delta P(\lambda) = \sum_{k=0}^m \lambda^k \Delta A_k = \left( \sum_{k=0}^m |\lambda|^k \alpha_k p(|\lambda|)^{-1} X \right) = X, \]
and \( \|\Delta A_k\| \leq \alpha_k \varepsilon, k = 0 : m \). Hence \( \lambda \in \Lambda_\varepsilon(P) \). \qed
We assume now that the matrices $\Delta A_k$ have a certain structure belonging to

$$\text{struct} \in \{\text{Toep}, \text{circ}, \text{Hankel}, \text{sym}\}. \quad (4.5)$$

We also suppose that all the matrices $A_k$ and $\Delta A_k$, $k = 0 : n$, belong to $M^\text{struct}_n(C)$ for a given structure in (4.5). Let

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M^\text{struct}_n(C)$, $k = 0 : m$ and

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M^\text{struct}_n(C)$. One notices that $P(\lambda)$ and $\Delta P(\lambda)$ belong to $M^\text{struct}_n(C)$. We define the structured $\varepsilon$-pseudospectrum of $P$ by

$$\Lambda^\text{struct}_\varepsilon(P) = \{\lambda \in \mathbb{C} : (P(\lambda) + \Delta P(\lambda)) x = 0 \text{ for some } x \neq 0 \text{ with } \Delta A_k \in M^\text{struct}_n(C), \|\Delta A_k\| \leq \alpha_k \varepsilon, k = 0 : n\}.$$

The following lemma is the structured version of Lemma 4.1.

**Lemma 4.2.** For struct in (4.5) we have

$$\Lambda^\text{struct}_\varepsilon(P) = \{\lambda \in \mathbb{C} : d^\text{struct}(P(\lambda)) \leq \varepsilon p(|\lambda|)\},$$

where $p(x) = \sum_{k=0}^n \alpha_k x^k$.

**Proof.** Let $\lambda$ in $\Lambda^\text{struct}_\varepsilon(P)$. It means that there exists $\Delta P(\lambda) \in M^\text{struct}_n(C)$ such that $\Delta A_k \in M^\text{struct}_n(C), \|\Delta A_k\| \leq \alpha_k \varepsilon, k = 0 : m$ and $P(\lambda) + \Delta P(\lambda)$ is singular. It follows from the definition of the distance $d^\text{struct}$ that $d^\text{struct}(P(\lambda)) \leq \|\Delta P(\lambda)\|$. Since

$$\|\Delta P(\lambda)\| \leq \sum_{k=0}^m |\lambda|^k \alpha_k \varepsilon = \varepsilon p(|\lambda|),$$

we obtain $d^\text{struct}(P(\lambda)) \leq \varepsilon p(|\lambda|)$.

Conversely, let $\lambda \in \mathbb{C}$ be such that $d^\text{struct}(P(\lambda)) \leq \varepsilon p(|\lambda|)$. It means that there exists $X \in M^\text{struct}_n(C)$ such that $\|X\| \leq \varepsilon p(|\lambda|)$ and $P(\lambda) + X$ singular. Let us define $\Delta A_k$ by

$$\Delta A_k = \text{sign}(\lambda^k) \alpha_k p(|\lambda|)^{-1} X.$$

Then, $\Delta A_k \in M^\text{struct}_n(C)$,

$$\Delta P(\lambda) = \sum_{k=0}^m \lambda^k \Delta A_k = \left(\sum_{k=0}^m |\lambda|^k \alpha_k p(|\lambda|)^{-1} X\right) = X,$$

and $\|\Delta A_k\| \leq \alpha_k \varepsilon$, $k = 0 : m$. Hence $\lambda \in \Lambda^\text{struct}_\varepsilon(P)$.

\[\square\]
From Lemma 4.1 and 4.2 and Theorem 2.1 we deduce the following theorem for struct in (4.5).

**Theorem 4.3.** Given \( \varepsilon > 0 \) and \( P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0 \) a matrix polynomial with \( A_k \in M_n^{\text{struct}}(\mathbb{C}) \), \( k = 0 : m \) for \( \text{struct} \in \{ \text{Toep}, \text{circ}, \text{Hankel}, \text{sym} \} \),

the \( \varepsilon \)-pseudospectrum and the structured \( \varepsilon \)-pseudospectrum satisfy

\[
\Lambda^\text{struct}_\varepsilon(P) = \Lambda_\varepsilon(P).
\]

### 4.2 Structured pseudospectra of real matrix polynomials

In this subsection, we consider that \( P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0 \),

with \( A_k \in M_n(\mathbb{R}) \), \( k = 0 : m \) and \( \Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0 \),

where \( \Delta A_k \in M_n(\mathbb{R}) \). We suppose that \( P(\lambda) \) is subject to structured perturbations that can be expressed as

\[
[\Delta A_0, \ldots, \Delta A_m] = D\Theta[E_0, \ldots, E_m],
\]

with \( D \in M_{n,1}(\mathbb{R}), \Theta \in M_{1,t}(\mathbb{R}) \) and \( E_k \in M_{t,n}(\mathbb{R}) \), \( k = 0 : m \). This type of structure arises naturally in control theory. For notational convenience, we introduce

\[
E(\lambda) = E[I_n, \lambda I_n, \ldots, \lambda^m I_n]^T = \lambda^m E_m + \lambda^{m-1} E_{m-1} + \cdots + E_0,
\]

and

\[
G(\lambda) = E(\lambda) P(\lambda)^{-1} D = G_R(\lambda) + iG_I(\lambda), \quad G_R(\lambda), G_I(\lambda) \in \mathbb{R}^t.
\]

We define the structured \( \varepsilon \)-pseudospectrum by

\[
\Lambda_\varepsilon(P) = \{ \lambda \in \mathbb{C} : (P(\lambda) + D\Theta E(\lambda))x = 0 \text{ for some } x \neq 0, \|\Theta\| \leq \varepsilon \}.
\]

In the sequel, we denote for \( x, y \in \mathbb{R}^t \),

\[
d(x, Ry) = \inf_{\alpha \in \mathbb{R}} \|x - \alpha y\|
\]

the distance of the point \( x \) from the linear subspace \( Ry = \{\alpha y, \alpha \in \mathbb{R} \} \). The following theorem provides a computable form of the structured pseudospectrum.
Theorem 4.4.

\[ \Lambda_{\epsilon}(P) = \{ \lambda \in \mathbb{C} \setminus \Lambda(P) : d(G_R(\lambda), RG_I(\lambda)) \geq 1/\epsilon \} \cup \Lambda(P) \]

**Proof.** If there exists \( x \neq 0 \) such that \( (P(\lambda) + D\Theta E(\lambda))x = 0 \) then \( x = -P(\lambda)^{-1}D\Theta E(\lambda)x \) so that \( \Theta E(\lambda)x = -\Theta E(\lambda)P(\lambda)^{-1}D\Theta E(\lambda)x \). Let us denote \( u = \Theta E(\lambda)x \in \mathbb{C}, u = u_1 + iv_2, (u_1, u_2) \in \mathbb{R}^2 \). It is clear that \( u \neq 0 \) since \( \lambda \notin \Lambda(P) \). Using these notations, we obtain

\[ u = -\Theta G(\lambda)u. \]

This can be rewritten in real terms by

\[
\begin{align*}
  u_1 &= -\Theta G_R(\lambda)u_1 + \Theta G_I(\lambda)u_2, \\
  u_2 &= -\Theta G_R(\lambda)u_2 - \Theta G_I(\lambda)u_1.
\end{align*}
\]

These equations are equivalent to

\[
\begin{align*}
  (1 + \Theta G_R(\lambda))u_1 - \Theta G_I(\lambda)u_2 &= 0, \\
  -\Theta G_I(\lambda)u_1 - (1 + \Theta G_R(\lambda))u_2 &= 0.
\end{align*}
\]

Since \((u_1, u_2) \neq (0, 0)\), the system has a non trivial solution. It follows that the determinant of the system vanishes. A simple calculation shows that this determinant equals \((1 + \Theta G_R(\lambda))^2 + (\Theta G_I(\lambda))^2\). We conclude that \( \Theta \) satisfies the above equations if and only if

\[ \Theta G_I(\lambda) = 0 \quad \text{and} \quad \Theta G_R(\lambda) = -1. \]

We deduce that for all \( \alpha \in \mathbb{R} \), we \( \Theta(G_R(\lambda) - \alpha G_I(\lambda)) = -1 \) so that we have \( 1 \leq \varepsilon \|G_R(\lambda) - \alpha G_I(\lambda)\| \). Hence we have

\[ d(G_R(\lambda), RG_I(\lambda)) \geq 1/\varepsilon. \]

Conversely, let us assume that \( d(G_R(\lambda), RG_I(\lambda)) \geq 1/\varepsilon \). From a duality theorem (see [6]) there exists a vector \( z \in \mathbb{R}^t, \|z\| = 1 \) such that

\[
\begin{align*}
  z^T G_R(\lambda) &= d(G_R(\lambda), RG_I(\lambda)), \\
  z^T G_I(\lambda) &= 0.
\end{align*}
\]

Let us define \( \Theta = -d(G_R(\lambda), RG_I(\lambda))^{-1}z \) and \( x = P(\lambda)^{-1}D \). In this case, we have \( (P(\lambda) + D\Theta E(\lambda))x = 0 \). \( \square \)
5 Conclusion

In this note we have shown that the structured pseudospectrum is equal to the pseudospectrum for the following structures: Toeplitz, circulant and symmetric. We have also shown that this result is false for structures Hermitian and skew-Hermitian. We have generalized these results to pseudospectra of matrix polynomials with Toeplitz, circulant, Hankel and symmetric structures. Moreover, we have given a formula for structured pseudospectra of real matrix polynomials.

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References


