Computation of pseudozero abscissa

Stef Graillat

Équipe de Recherche DALI
Laboratoire LP2A, Université de Perpignan
52, avenue Paul Alduy, F-66860 Perpignan Cedex, France
graillat@univ-perp.fr
http://gala.univ-perp.fr/~graillat

Abstract. A polynomial p is robustly stable when all the zeros of the complex polynomials within a given distance of p lie in the left half-plane. The pseudozero abscissa, which is the largest real part of those zeros, measures the robust stability of p. Three algorithms to compute the pseudozero abscissa are presented. The first one is a graphical tool, the second one is a bisection algorithm whereas the third one is a criss-cross algorithm.

AMS Subject Classification: 65F35, 68W30
Keywords and phrases: pseudozero, abscissa mapping, bisection algorithm, crisscross algorithm

1 Introduction

In control theory, classical transfer functions associated with some systems are often written as $H(z) = N(z)/D(z)$, where N and D are complex polynomials and z is a parameter of the system. The system described with the function H is stable (in the sense of Hurwitz) if the polynomial D is stable, that is if all the zeros of D have negative real part. Since uncertainties on the coefficients of the polynomials are unavoidable in most real problems (data uncertainty, rounding error), it is useful to know if the system still remains stable when the polynomial coefficients suffer from an uncertainty of $\varepsilon > 0$. In such cases, the system is robustly stable. Similarly the polynomial D is robustly stable when all the zeros of the complex polynomials within a given distance $\varepsilon > 0$ of p lie in the left half-plane. The largest real part of these zeros measures the robust stability of the polynomial p and is defined to be the pseudozero abscissa of p. This paper focuses on this pseudozero abscissa.

Using a companion matrix, this polynomial problem could be reformulated as a matrix problem. A matrix $A \in \mathbb{C}^{n\times n}$ is stable if all its eigenvalues have a negative real part, and unstable otherwise. When A is stable, it may be interesting to know if the matrix remains stable after small perturbations on the its coefficients. We say that A is robustly stable if all eigenvalues of complex matrices within a given distance of A lie in the left half-plane. The pseudospectra abscissa of A is the largest real part of such eigenvalues; it measures the robust stability of A. This quantity has been studied in numerical linear algebra [3, 4]. To ensure that the pseudospectra abscissa of the companion matrix A is the associated pseudozero abscissa of the polynomial p, the perturbed matrix $A+E$ has to conserve the companion structure of A. Up to our knowledge, no existing matrix algorithm guarantees this property. Moreover, it is not worth to transform a polynomial problem into a matrix problem which increases the complexity of the problem.

In this paper, we propose three algorithms that compute the polynomial pseudozero abscissa staying in the field of polynomials. In these three algorithms, the key tool to succeed is the polynomial pseudozero set introduced by Mosier [13]. The first algorithm is a graphical use of pseudozero set plots. The second algorithm is a symbolic-numeric algorithm (sometimes called hybrid algorithm, see [9]) based on a bisection. It means that we use algebraic techniques (here Sturm sequences) within a numerical procedure. This approach has been previously proposed in [2] to count the number of imaginary eigenvalues of an Hamiltonian matrix. It seems that it has not received much more attention whereas it can provide efficient and accurate algorithms. The third one is a criss-cross strategy. It optimizes the previous bisection step moving alternatively along specific horizontal and vertical lines in the pseudozero set.

The paper is organized as follows. In Section 2 and 3, we present definitions and useful results about pseudozero sets. In Section 4, we propose three algorithms to compute the pseudozero abscissa.

2 Definition and computation of pseudozero set

In this section, we present some results about pseudozero set. In the first subsection, we define the notion of pseudozero set. In the second subsection, we provide a computable formula for the pseudozero set. In the last subsection, we present an algorithm to compute it.

2.1 Definition of the ε-pseudozero set

For $n \geq 1$, let $\mathcal{P}_n$ be the linear space of polynomials of degree at most n with complex coefficients and $\mathcal{M}_n$ the subset of monic polynomials of degree n. Let $p \in \mathcal{M}_n$ be given by

$$p(z) = \sum_{i=0}^{n} p_i z^i, \quad p_n = 1. \tag{1}$$

Representing p by the vector $(p_0, \ldots, p_{n-1})^T$ of its coefficients, we define the norm $\| \cdot \|$ on $\mathcal{M}_n$ as the 2-norm on $\mathbb{C}^n$ of the corresponding vector. It means that

$$\|p\| = \left(\sum_{i=0}^{n-1} |p_i|^2 \right)^{1/2}.$$

We recall the notion of pseudozero set and we give a useful computable characterization. These results can be found in [13, 15, 16].

Given a real $\varepsilon > 0$, the ε-neighborhood of $p \in \mathcal{M}_n$ is the set

$$N_\varepsilon(p) = \{\widehat{p} \in \mathcal{M}_n : \|p - \widehat{p}\| \le \varepsilon\}\,. \tag{2}$$

The ε-pseudozero set of p is the set of all the zeros of the ε-neighborhood, that is to say,

$$Z_\varepsilon(p) = \{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\}\,. \tag{3}$$

2.2 A computable form of the ε-pseudozero set

One has the following characterization of the ε-pseudozero set.

Theorem 1 (Trefethen and Toh [16]). *The ε-pseudozero set satisfies*

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : g(z) := \frac{|p(z)|}{\|\underline{z}\|} \le \varepsilon \right\}, \tag{4}$$

where $\underline{z} = (1, z, \ldots, z^{n-1})^T$.

This theorem was first proved in [16] for the 2-norm. A proof for an arbitrary norm can be found in [15, 10]. We recall the proof of [16] for completeness.

Proof. If $z \in Z_\varepsilon(p)$ then there exists $\widehat{p} \in \mathcal{M}_n$ such that $\widehat{p}(z) = 0$ and $\|p - \widehat{p}\| \le \varepsilon$. From Hölder's inequality $|x^T y| \le \|x\|\|y\|$, we get

$$|p(z)| = |p(z) - \widehat{p}(z)| = |\sum_{i=0}^{n} (p_i - \widehat{p}_i) z^i| \le \|p - \widehat{p}\|\|\underline{z}\|.$$

It follows $|p(z)| \le \varepsilon\|\underline{z}\|$.

Conversely, let $u \in \mathbb{C}$ be such that $|p(u)| \le \varepsilon\|\underline{u}\|$. Let us write $u = |u|e^{i\theta}$. Let us introduce the polynomials r and p_u defined by

$$r(z) = \sum_{k=0}^{n-1} r_k z^k \quad \text{with} \quad r_k = |u|^k e^{-ik\theta}, \tag{5}$$

$$p_u(z) = p(z) - \frac{p(u)}{r(u)} r(z). \tag{6}$$

It is clear that $r(u) = \|\underline{u}\|^2 = \|r\|^2$, and $p_u(u) = 0$. So we have

$$\|p - p_u\| = \frac{|p(u)|}{|r(u)|}\|r\| \le \varepsilon,$$

hence $u \in Z_\varepsilon(p)$. □

2.3 Computing the ε-pseudozero set

Theorem 1 yields a computable expression for the ε-pseudozero set. It consists in evaluating the easily computable function g on a grid of the complex plane and comparing its value to the ε parameter.

MATLAB software, for example, provides primitives that allow us to plot pseudozeros with the following very simple Algorithm 1. Such an implementation is very similar to existing pseudospectra software [7].

Algorithm 1 Computation of ε-pseudozero set

Require: polynomial p and precision ε

Ensure: pseudozero set layout in the complex plane

1: We grid a square containing all the roots of p with the MATLAB command `meshgrid`.

2: We compute $g(z)$ for the whole points z on the grid.

3: We draw the level line $|g(z)| = \varepsilon$ with the MATLAB command `contour`.

3 Topological and geometric properties of pseudozero set

In this section, we establish that the pseudozero set is a compact set and that the closure of the strict pseudozero set is the pseudozero set. The *strict ε-pseudozero set* is defined to be

$$Z'_\varepsilon(p) = \{z \in \mathbb{C} : q(z) = 0 \text{ where } \|p - q\| < \varepsilon\}.$$

With the same proof as in Theorem 1, it follows that

$$Z'_\varepsilon(p) = \{z \in \mathbb{C} : g(z) < \varepsilon\}, \tag{7}$$

where

$$g(z) = \frac{|p(z)|}{\|\underline{z}\|}, \quad \|\underline{z}\| = \|(1, z, \dots, z^{n-1})\|.$$

Let the function $h : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$h_\varepsilon(x, y) = |p(x + iy)|^2 - \varepsilon^2 \sum_{j=0}^{n-1} (x^2 + y^2)^j. \tag{8}$$

For a fixed x_0, the function $h_\varepsilon(x_0, \cdot)$ is a polynomial of degree $2n$. In the same way, for a fixed y_0, the function $h_\varepsilon(\cdot, y_0)$ is a polynomial of degree $2n$. From Theorem 1, the ε-pseudozero set satisfies

$$Z_\varepsilon(p) = \{(x, y) \in \mathbb{R}^2 : h_\varepsilon(x, y) \le 0\}.$$

Theorem 2. *The ε-pseudozero set of $p \in \mathcal{M}_n$ is a compact set contained in the ball of radius $1 + \|p\| + \varepsilon$.*

Proof. As the function h is continuous, the set $Z_\varepsilon(p) = h^{-1}((-\infty, \varepsilon])$ is closed. Let us denote by $\{z_j\}_{j=1:n}$ the roots of the polynomial p and $r = \max_j |z_j|$. It is well-known (see [12] for example) that

$$r \le 1 + \|p\|.$$

If $z \in Z_\varepsilon(p)$ then there exists $\widehat{p} \in \mathcal{P}_n$ satisfying both both $\widehat{p}(z) = 0$ and $\|p - \widehat{p}\| \le \varepsilon$. It follows that $|z| \le 1 + \|\widehat{p}\|$. Furthermore, we have $|\|\widehat{p}\| - \|p\|| \le \|\widehat{p} - p\| \le \varepsilon$ and so $\|\widehat{p}\| \le \|p\| + \varepsilon$. Hence $|z| \le 1 + \|p\| + \varepsilon$. $\square$

The following theorem shows that each connected component of $Z_\varepsilon(p)$ contains at least one root of $p \in \mathcal{M}_n$. It extends Theorem 2 from Mosier [13] for monic polynomials and the 2-norm.

Theorem 3. *If $q \in N_\varepsilon(p)$, then p and q have the same number of roots, counting multiplicities, in each connected component of $Z_\varepsilon(p)$. Furthermore, there is at least one root of p in each connected component of $Z_\varepsilon(p)$.*

Proof. Let us define the monic polynomials $p_t(z) = (1-t)p(z) + tq(z)$, $t \in [0, 1]$. We have $\|p - p_t\| = t\|p - q\|$, $t \in [0, 1]$, so p_t belongs to $N_\varepsilon(p)$. This implies that all roots of p_t lie in $Z_\varepsilon(p)$. As the roots depend continuously of the coefficients of the polynomial, when t varies from 0 to 1, the roots of p_t trace continuous paths from the roots of p to the roots of q. Since the connected components are bounded and disjoint, no root can move to another component nor disappear. Thus all the polynomials p_t have the same number of roots in each connected component.
Now we prove the second assertion. Let z belonging to a connected component of $Z_\varepsilon(p)$. By definition, there exists a polynomial $\widehat{p} \in N_\varepsilon(p)$ such that $\widehat{p}(z) = 0$. Since p has the same number of roots as $\widehat{p}$ in the connected component, it follows that p has at least a root in this component. □

Theorem 4. *The closure of the strict ε-pseudozero set is the ε-pseudozero set.*

Proof. As $Z'_\varepsilon(p) \subset Z_\varepsilon(p)$ and $Z_\varepsilon(p)$ is closed, it is clear that $\overline{Z'_\varepsilon(p)} \subset Z_\varepsilon(p)$. Now, let $z \in Z_\varepsilon(p)$ be such that $g(z) = \varepsilon$. Thus there exists $q \in N_\varepsilon(p)$ satisfying $q(z) = 0$ and $\|p-q\| = \varepsilon$. Let us define the polynomial $p_t(z) = (1-t)p(z)+tq(z)$, $t \in [0, 1]$. We clearly have $p_0(z) = p(z)$ and $p_1(z) = q(z)$. Moreover, $\|p_t - p\| = t\|p - q\| = t\varepsilon < \varepsilon$ for all $t \in [0, 1)$. Since the roots depend continuously of the coefficients of the polynomial there exist n continuous function $t \mapsto z_i(t)$, $i = 1 : n$ that represent the n roots of p_t. Consequently, we have $p_t(z_i(t)) = 0$ for $i = 1 : n$ and $t \in [0, 1]$. By taking limit as $t \to 1$, we obtain $p_1(z_i(1)) = q((z_i(1)) = 0$. Hence, $z_j(1) = z$ for some j in $\{1, \dots, n\}$. Taking into account that $\|p_t - p\| < \varepsilon$ and $p_t(z_j(t)) = 0$, we conclude that $z_j(t) \in Z'_\varepsilon(p)$ for $t \in [0, 1)$. This ends the proof. □

4 Pseudozero abscissa

In this section, we define the pseudozero abscissa and provide three algorithms to compute this quantity. The first one is a qualitative algorithm based on drawing a pseudozero set. The second one is based on a bisection algorithm. The third one is based on a criss-cross algorithm with vertical and horizontal searches.

As in [6], we define the *abscissa mapping* $a : \mathcal{P}_n \to \mathbb{R}$ as follows,

$$a(p) = \max\{\mathrm{Re}(z) : p(z) = 0\}.$$

Our first aim is to prove that the abscissa mapping a is continuous on $\mathcal{M}_n$. It is clear that a is not continuous on $\mathcal{P}_n$ as it is proved in [5]. Indeed, let us consider the polynomial $q_t(z) = (1 - tz)p(z)$, where p is a polynomial of degree at most $n-1$. We have $q_t \to p$ when $t \to 0$, whereas $a(q_t) = 1/t$, that is arbitrary larger than $a(p)$.

To prove the continuity of a on $\mathcal{M}_n$, we will use the following result known as "the continuous dependence of the zeros of a polynomial on its coefficients". The proof can be found in [11, 14]. In the following proposition, $\mathfrak{S}_n$ is the symmetric group of degree n.

Proposition 1 (Ostrowski [14]). *Let*

$$p(z) = p_0 + p_1 z + \cdots + p_{n-1} z^{n-1} + z^n$$

be a monic polynomial with complex coefficients. Then, for every $\varepsilon > 0$, there is $\eta > 0$ such that for any polynomial

$$q(z) = q_0 + q_1 z + \cdots + q_{n-1} z^{n-1} + z^n$$

satisfying

$$\max_{0 \leq i \leq n} |p_i - q_i| < \eta,$$

we have

$$\min_{\sigma \in \mathfrak{S}_n} \max_{1 \leq j \leq n} |x_j - y_{\sigma(j)}| < \varepsilon,$$

where (x_j) and (y_j), $j = 1, \ldots, n$, are respectively the zeros of p and q.

We can now prove the continuity of a on $\mathcal{M}_n$.

Proposition 2. *The abscissa mapping*

$$a : \mathcal{P}_n \to \mathbb{R},$$

defined by $a(p) = \max\{\mathrm{Re}(z) : p(z) = 0\}$ is continuous on $\mathcal{M}_n$.

Proof. Let p in $\mathcal{M}_n$ and $\varepsilon > 0$. From Proposition 1, there is $\eta > 0$ such that for any q in $\mathcal{M}_n$ satisfying

$$\max_{0 \leq i \leq n} |p_i - q_i| < \eta,$$

we have

$$\min_{\sigma \in \mathfrak{S}_n} \max_{1 \leq j \leq n} |x_j - y_{\sigma(j)}| < \varepsilon,$$

where (x_j) and (y_j), $j = 1, \ldots, n$, are respectively the zeros of p and q. This means that there is a permutation σ in $\mathfrak{S}_n$ such that

$$\max_{1 \leq j \leq n} |\mathrm{Re}(x_j) - \mathrm{Re}(y_{\sigma(j)})| \leq \max_{1 \leq j \leq n} |x_j - y_{\sigma(j)}| < \varepsilon.$$

We have

$$\begin{aligned}
|a(q) - a(p)| &= |\max_{1 \leq j \leq n} \mathrm{Re}(y_j) - \max_{1 \leq j \leq n} \mathrm{Re}(x_j)| \\
&= |\max_{1 \leq j \leq n} \mathrm{Re}(y_{\sigma(j)}) - \max_{1 \leq j \leq n} \mathrm{Re}(x_j)| \\
&\leq \max_{1 \leq j \leq n} |\mathrm{Re}(y_{\sigma(j)}) - \mathrm{Re}(x_j)| \\
&\leq \varepsilon.
\end{aligned}$$

We have proved the continuity of a on $\mathcal{M}_n$. $\square$

A natural extension of the abscissa mapping when polynomials are perturbed is the *ε-pseudozero abscissa mapping* $a_\varepsilon : \mathcal{P}_n \to \mathbb{R}$ defined by

$$a_\varepsilon(p) = \max\{\mathrm{Re}(z) : z \in Z_\varepsilon(p)\}. \tag{9}$$

The ε-pseudozero abscissa is the maximum value of the real part over the ε-pseudozero set. From Theorem 4, we deduce that the ε-pseudozero abscissa satisfies

$$a_\varepsilon(p) = \sup\{\mathrm{Re}(z) : z \in Z'_\varepsilon(p)\}.$$

We can introduce the function h_ε defined in (8) in definition (9) of the ε-pseudozero abscissa since

$$a_\varepsilon(p) = \sup\{x : (x,y) \in \mathbb{R}^2,\ h_\varepsilon(x,y) \le 0\}.$$

In the sequel, we propose to compute the ε-pseudozero abscissa presenting three algorithms.

4.1 Drawing pseudozero set

A first way to compute the pseudozero abscissa is to draw the ε-pseudozero set using Algorithm 1. Once one has drawn this set, it suffices to draw the vertical line that intersects the right-most point within the ε-pseudozero. Thus the pseudozero abscissa is the real value being the intersection between this vertical line and the real axis.

Let us choose for example $p(z) = z^3 + 4z^2 + 6z + 4$ with $\varepsilon = 0.1$. We draw the ε-pseudozero set (see Figure 1) and identify that $a_\varepsilon(p) \approx -0.9$.

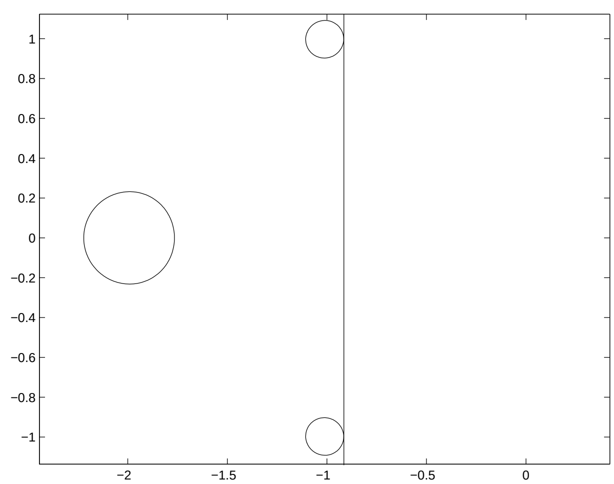

Fig. 1. Pseudozero set of $p(z) = z^3 + 4z^2 + 6z + 4$ with $\varepsilon = 0.1$

Plotting pseudozero set gives qualitative and quantitative interesting informations about the robust stability of polynomials. They can be easily plotted using popular software as MATLAB. Other graphical algorithms relying on pseudozero set have been proposed in [10].

4.2 A bisection algorithm

In this subsection we introduce a bisection algorithm to compute $a_\varepsilon(p)$ to an arbitrary accuracy. The key step to derive this algorithm is Theorem 5 below. Before stating this theorem, we need the following lemma.

Lemma 1. *For any point z_1 in $Z_\varepsilon(p)$, there exists a point z_2 satisfying* $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ *and* $h_\varepsilon(z_2) = 0$.

Proof. We can take z_2 on the boundary of the intersection of the vertical line through z_1 and the pseudozero set $Z_\varepsilon(p)$ (it is a compact set). $\square$

The main result in this subsection is the following theorem. It yields a computable property of lower bounds of $a_\varepsilon(p)$.

Theorem 5. *For any real $x \geq a(p)$, the following statements are equivalent:*

(i) $x \leq a_\varepsilon(p)$;
(ii) *the polynomial equation*

$$h_\varepsilon(x, y) = 0, \tag{10}$$

admits a real solution y.

Proof. We first show (i) $\Rightarrow$ (ii). If $x = a_\varepsilon(p)$ then we choose a point z solving the pseudozero abscissa problem, that is to say such that $\mathrm{Re}(z) = a_\varepsilon(p)$. From Lemma 1, it is clear that $z = x+iy$ for some real y and $h_\varepsilon(x, y) = 0$. If $x < a_\varepsilon(p)$, by definition of $a_\varepsilon(p)$, there exists a point z_1 such that $\mathrm{Re}(z_1) > x$ and $g(z_1) < \varepsilon$. The connected component of z_1 in $Z_\varepsilon(p)$ contains a root z_2 of p by Theorem 3. Therefore, there is a continuous arc in the component connecting z_1 and z_2 (see the proof of Theorem 3). But since $\mathrm{Re}(z_1) \geq x > \mathrm{Re}(z_2)$, this arc must contain a point z_3 such that $\mathrm{Re}(z_3) = x$. From Lemma 1, we get a solution to equation (10).
We now prove (ii) $\Rightarrow$ (i). Let $y \in \mathbb{R}$ be such that $h_\varepsilon(x, y) = 0$. This implies that $x + iy \in Z_\varepsilon(p)$. Using the definition of $a_\varepsilon(p)$, we derive $x \leq a_\varepsilon(p)$.

Now we can explain the bisection algorithm that yields an approximation of $a_\varepsilon(p)$. From the definition of $a_\varepsilon(p)$ and Theorem 2, we know that $a_\varepsilon(p)$ lies in the interval $[a(p), 1 + \|p\| + \varepsilon]$. Let us denote x the midpoint of this interval. We compute the solution of equation $h_\varepsilon(x, y) = 0$, $y \in \mathbb{R}$. If one of the roots is real then we deduce that $x \leq a_\varepsilon(p)$ by Theorem 5, and we replace the current interval by $[x, 1+\|p\|+\varepsilon]$. Otherwise, if $x > a_\varepsilon(p)$, we replace the current interval by $[a(p), x]$. The implementation of this method is proposed with Algorithm 2 where τ denotes the requested accuracy for the approximation of $a_\varepsilon(p)$.

Algorithm 2 Computation of ε-pseudozero abscissa by bisection

Require: a polynomial p, the parameter ε and a tolerance τ
Ensure: a number α such that $|\alpha - a_\varepsilon(p)| \le \tau$

1: $\gamma := a(p), \quad \delta := 1 + \|p\| + \varepsilon$
2: **while** $|\gamma - \delta| > \tau$ **do**
3: $\quad x := \frac{\gamma+\delta}{2}$
4: $\quad$ **if** the equation $h_\varepsilon(x, y) = 0,\ y \in \mathbb{C}$ has a real solution **then**
5: $\qquad \gamma := x$
6: $\quad$ **else**
7: $\qquad \delta := x$
8: $\quad$ **end if**
9: **end while**
10: **return** $\alpha = \frac{\gamma+\delta}{2}$

The difficult step of this algorithm is to test whether the polynomial $H_x(y) = h_{p,\varepsilon}(x, y) = h_{2n}y^{2n} + \cdots + h_1 y + h_0$ has real roots. We recall that H_x has real coefficients because of the definition of h_ε. We describe how to solve this question using Sturm sequences. It is well-known that the maximum modulus r of the zeros of H_x satisfies (see [12])

$$r \le \frac{\|H_x\|}{|h_{2n}|}.$$

Consequently, the possible real roots of H_x belong necessarily to the interval $[-r, r]$. One may apply the Euclid's algorithm to the polynomial H_x and its derivative H_x'. Denote $H_0 = H_x$ and $H_1 = H_x'$ and define

$$H_{i+1} = -\operatorname{rem}(H_{i-1}, H_i),$$

which is the remainder of the division of H_{i-1} by H_i. Let m be the smallest integer such that $H_{m+1} = 0$. Let $v_H(-r)$ be the number of sign changes in the leading coefficients of $H_0(-r), \ldots, H_m(-r)$ and let $v_H(r)$ be the number of sign changes in the leading coefficients of $H_0(r), \ldots, H_m(r)$. So we have defined a Sturm sequence and Sturm's Theorem ensures that H_x has exactly $v_H(-r) - v_H(r)$ distinct real roots (see [12]). In particular H_x has a real root if and only if $v_H(-r) \neq v_H(r)$. Let us remark that Sturm sequences suffice to answer to the line 4 of Algorithm 2 without having to compute all the roots of $H_x(y) = h_\varepsilon(x, y)$. Instead of Strum sequences, we could use Sturm-Habicht sequences (see [8]). Since Sturm-Habicht sequences deal with real roots in a generic way, this would avoid to compute Sturm sequences at each step of the algorithm.

Algorithm 2 is implemented using the MAPLE software. This choice is natural since we need some algebraic manipulations of polynomials.

Let us take the same example as in Subsection 4.1, that is, $p(z) = z^3 + 4z^2 + 6z + 4$ with $\varepsilon = 0.1$ and $\tau = 0.00001$. We find that $a_\varepsilon(p) \approx -0.919901$ which is more precise that the graphical result given previously.

Let us comment another example choosing $q(z) = z^5 + 5z^4 + 10z^3 + 10z^2 + 5z + 1$ and $\varepsilon = 0.001$. Algorithm 2 gives $a_\varepsilon(q) = -0.719669$. Figure 2 represents the ε-pseudozero set of q for $\varepsilon = 0.001$, $\tau = 0.00001$ and gives a graphical representation of $a_\varepsilon(p)$.

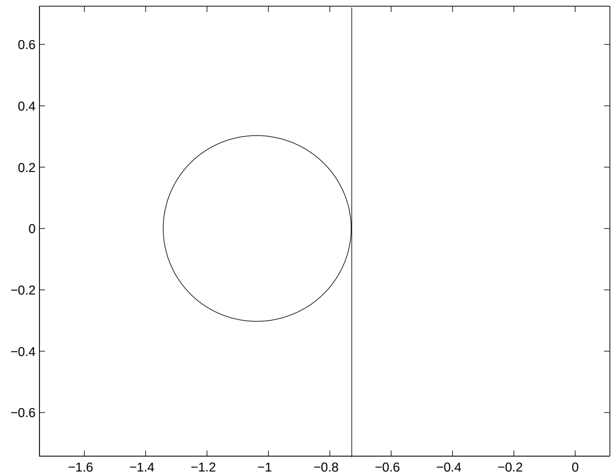

Fig. 2. Pseudozero set of $q(z) = z^5 + 5z^4 + 10z^3 + 10z^2 + 5z + 1$ with $\varepsilon = 0.001$

4.3 A criss-cross algorithm

We present in this last subsection a criss-cross algorithm inspired from [1, 4]. We begin with the following result.

Lemma 2. *For any $x \in (a(p), a_\varepsilon(p))$, there exists $y \in \mathbb{R}$ such that $h_\varepsilon(x, y) < 0$.*

Proof. Let z_0 be a root of p such that $\mathrm{Re}(z_0) = a(p)$ and $z_1 \in Z_\varepsilon(p)$ be such that $\mathrm{Re}(z_1) = a_\varepsilon(p)$. Let $q \in \mathcal{M}_n$ such that $q(z_1) = 0$ and $\|p - q\| \leq \varepsilon$. Let us define the polynomial $p_t(z) = (1-t)p(z) + tq(z)$, $t \in [0, 1]$. We clearly have $p_t \in \mathcal{M}_n$, $p_0(z) = p(z)$ and $p_1(z) = q(z)$. Let us introduce $\varphi : [0, 1] \to \mathbb{R}$ be the function $t \mapsto a(p_t)$. The continuity of a on $\mathcal{M}_n$ (see Proposition 2) ensures that the function φ is continuous. As $\varphi(0) = a(p)$ and $\varphi(1) \geq a_\varepsilon(p)$, there exists $\overline{t} \in (0, 1)$ such that $a(p_{\overline{t}}) = x$. As $\|p - p_{\overline{t}}\| < \varepsilon$, at least one root of $p_{\overline{t}}$ with real part x lies in $Z'_\varepsilon(p)$, hence $h(x, y) < 0$. □

In the following theorem, we prove the convergence of Algorithm 3.

Theorem 6. *The criss-cross algorithm converges to the pseudozero abscissa $a_\varepsilon(p)$.*

Proof (Sketch of proof). The proof follows the ideas of the proof of Theorem 3.2 in [4]. Therefore we will only give an outline of the proof. We denote $I_i^r = (l_i^r, u_i^r)$.

Algorithm 3 Computation of ε-pseudozero abscissa by criss-cross

Require: a polynomial p, the parameter ε

1: *Initialize:* $x^1 = a(p)$ and $r = 1$
2: *Vertical search:* find open intervals $I_1^r, \dots, I_{l_r}^r$ where $h(x^r, y) < 0$ for $y \in \cup_{k=1}^{l_r} I_k^r$
3: *Horizontal search:* for each I_k^r, define $\omega_k^r = \text{midpoint}(I_k^r)$ and find the largest real zeros x_k^r of the function $h(\cdot, \omega_k^r)$ for $k = 1 : l_r$
4: Define $x^{r+1} = \max\{x_k^r, k = 1, \dots, l_r\}$, increment r by one and return to Step 2.

The fact that the new iterate x^{r+1} is a zero of $h(\cdot, \omega_k^r)$ means that $x^{r+1} \leq a_\varepsilon(p)$. Since $x^1 = a(p) \leq a_\varepsilon(p)$, we prove by induction that $x^r \leq a_\varepsilon(p)$ for all r. If at iteration r, $x^r = a_\varepsilon(p)$, there is nothing to do. Otherwise, there exists j such that $l_j^r < u_j^r$ and so $h_\varepsilon(x^r, \omega_j^r) < 0$. It follows that $x^{r+1} > x^r$. So we deduce that the sequence (x^r) is increasing and bounded above strictly by $a_\varepsilon(p)$ and bounded below by $a(p)$. It follows that the sequence (x^r) converges to a real number x^∞ less or equal to $a_\varepsilon(p)$. Let us suppose that $x^\infty < a_\varepsilon(p)$. In this case, from Lemma 2, we derive that there exists an open interval in which we have $h_\varepsilon(x^\infty, y) < 0$. It follows that one can find ω_k^r for sufficiently large r and k such that $h_\varepsilon(x^\infty, \omega_k^r) \leq 0$. This implies that $x^{r+1} \geq x^\infty$. This contradicts the fact that the sequence (x^r) is strictly increasing. Hence we have $x^\infty = a_\varepsilon(p)$. □

5 Conclusion

In this paper, we have presented three algorithms to compute the pseudozero abscissa. The first one (drawing pseudozero set) is a qualitative algorithm based on plots. It seems to be the least efficient but it may be quite useful if we want to visualize the situation. The two next ones are symbolic-numeric algorithms that use analytical properties of pseudozero set. One is a bisection algorithm and the other one is a criss-cross algorithm. The latter seems to be the most efficient but needs the computation of polynomial roots which is expensive. Future work needs to be done on this algorithm.

References

1. S. Boyd and V. Balakrishnan. A regularity result for the singular values of a transfer matrix and a quadratically convergent algorithm for computing its L_∞-norm. *Syst. Control Lett.*, 15(1):1–7, 1990.
2. S. Boyd, V. Balakrishnan, and P. Kabamba. A bisection method for computing the H_∞ norm of a transfer matrix and related problems. *Math. Control Signals Syst.*, 2(3):207–219, 1989.
3. James V. Burke, Adrian S. Lewis, and Michael L. Overton. Optimization and pseudospectra, with applications to robust stability. *SIAM J. Matrix Anal. Appl.*, 25(1):80–104, 2003.
4. James V. Burke, Adrian S. Lewis, and Michael L. Overton. Robust stability and a criss-cross algorithm for pseudospectra. *IMA J. Numer. Anal.*, 23(3):359–375, 2003.

5. James V. Burke, Adrian S. Lewis, and Michael L. Overton. Variational analysis of the abscissa mapping for polynomials via de Gauss-Lucas theorem . *J. Global Optim.*, 28(3-4):259–268, 2004.
6. James V. Burke and Michael L. Overton. Variational analysis of the abscissa mapping for polynomials. *SIAM J. Control Optimization*, 39(6):1651–1676, 2001.
7. Mark Embree and Nick Trefethen. *Pseudospectra Gateway*, 2004. http://web.comlab.ox.ac.uk/projects/pseudospectra/.
8. Laureano Gonzalez-Vega and Ioana Necula. Efficient topology determination of implicitly defined algebraic plane curves. *Comput. Aided Geom. Design*, 19(9):719–743, 2002.
9. Johannes Grabmeier and Erich Kaltofen, editors. *Computer algebra handbook*. Springer-Verlag, Berlin, 2003.
10. Stef Graillat and Philippe Langlois. Testing polynomial primality with pseudozeros. In *Proceedings of the Fifth Conference on Real Numbers and Computers, Lyon, France*, pages 231–246, Lyon, France, September 2003.
11. Roger A. Horn and Charles R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990.
12. Maurice Mignotte. *Mathematics for computer algebra*. Springer-Verlag, New York, 1992.
13. Ronald G. Mosier. Root neighborhoods of a polynomial. *Math. Comp.*, 47(175):265–273, 1986.
14. A.M. Ostrowski. *Solution of equations and systems of equations.* Pure and Applied Mathematics. Vol. 9. 2nd ed. New York and London: Academic Press. XIV, 338 p., 1966.
15. Hans J. Stetter. *Numerical polynomial algebra*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2004.
16. Kim-Chuan Toh and Lloyd N. Trefethen. Pseudozeros of polynomials and pseudospectra of companion matrices. *Numer. Math.*, 68(3):403–425, 1994.