Validated Pseudozero Set of Polynomials

Stef Graillat
Université Pierre et Marie Curie - Paris 6, LIP6

stef.graillat@lip6.fr
http://www-anp.lip6.fr/~graillat

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Outline of the talk

I — Pseudozero set of polynomials

- Definition
- Computation

II — Validation of pseudozero set

- Validated computation
- Drawing of pseudozero set

Pseudozeros : definition, computation and motivation

Pseudozero set : definition

Perturbation :

Neighborhood of polynomial p

$$N_\varepsilon(p) = \{\widehat{p} \in \mathbb{C}_n[z] : \|p - \widehat{p}\| \leqslant \varepsilon\}\,.$$

Definition of the ε-pseudozero set :

$$Z_\varepsilon(p) = \{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\}\,.$$

$\|\cdot\|$ a norm on the vector of the coefficients of p

This set is formed by the zeros of polynomials "near p".

Pseudozeros are easily computable

Theorem :
The ε-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|\underline{z}\|_*} \leqslant \varepsilon \right\},$$

where $\underline{z} = (1, z, \dots, z^n)$ and $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$,

$$\|y\|_* = \sup_{x \neq 0} \frac{|y^* x|}{\|x\|}$$

Algorithm of computation

Algorithm to draw the ε-pseudozero set :

1. We mesh a square containing all the roots of p (MATLAB command : `meshgrid`).
2. We compute $g(z) := \frac{|p(z)|}{\|\underline{z}\|_*}$ for all the nodes z in the grid.
3. We draw the contour level $|g(z)| = \varepsilon$ (MATLAB command : `contour`).

Problems :

- Find a square containing all the roots of p and all the pseudozeros.
- we may evaluate p near some roots ! ! !

A famous example

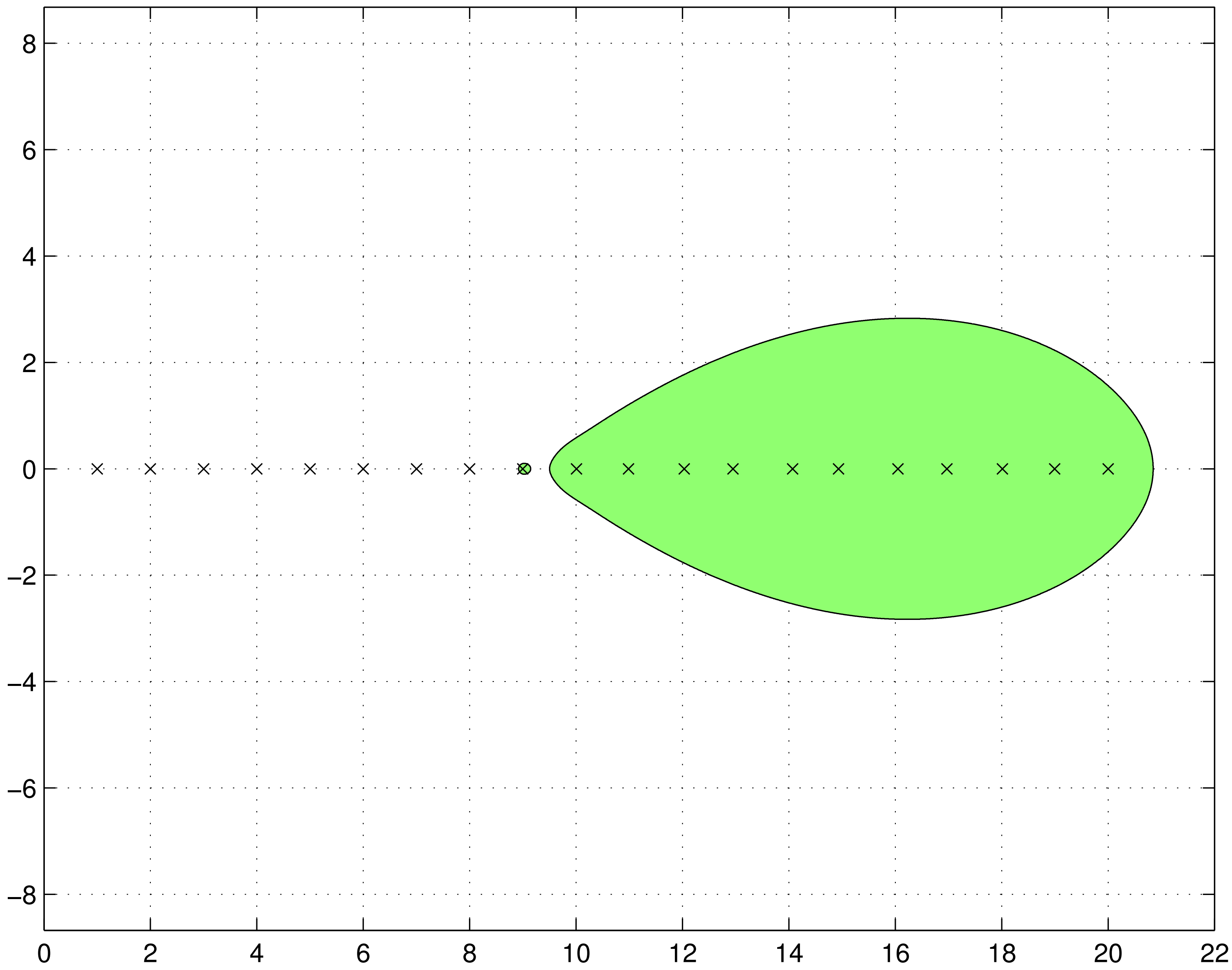
Pseudozero set of the *Wilkinson* polynomial

$$\begin{aligned} W_{20} &= (z-1)(z-2)\cdots(z-20), \\ &= z^{20} - 210z^{19} + \cdots + 20!. \end{aligned}$$

We perturb only the coefficient of z^{19} with $\varepsilon = 2^{-23}$.
One use the weighted-norm $\|\cdot\|_\infty$:

$$\|p\|_\infty = \max_i \frac{|p_i|}{m_i} \text{ with } m_i \text{ non negative}$$

with $m_{19} = 1$, $m_i = 0$ otherwise and the convention $m/0 = \infty$ if $m > 0$ and $0/0 = 0$.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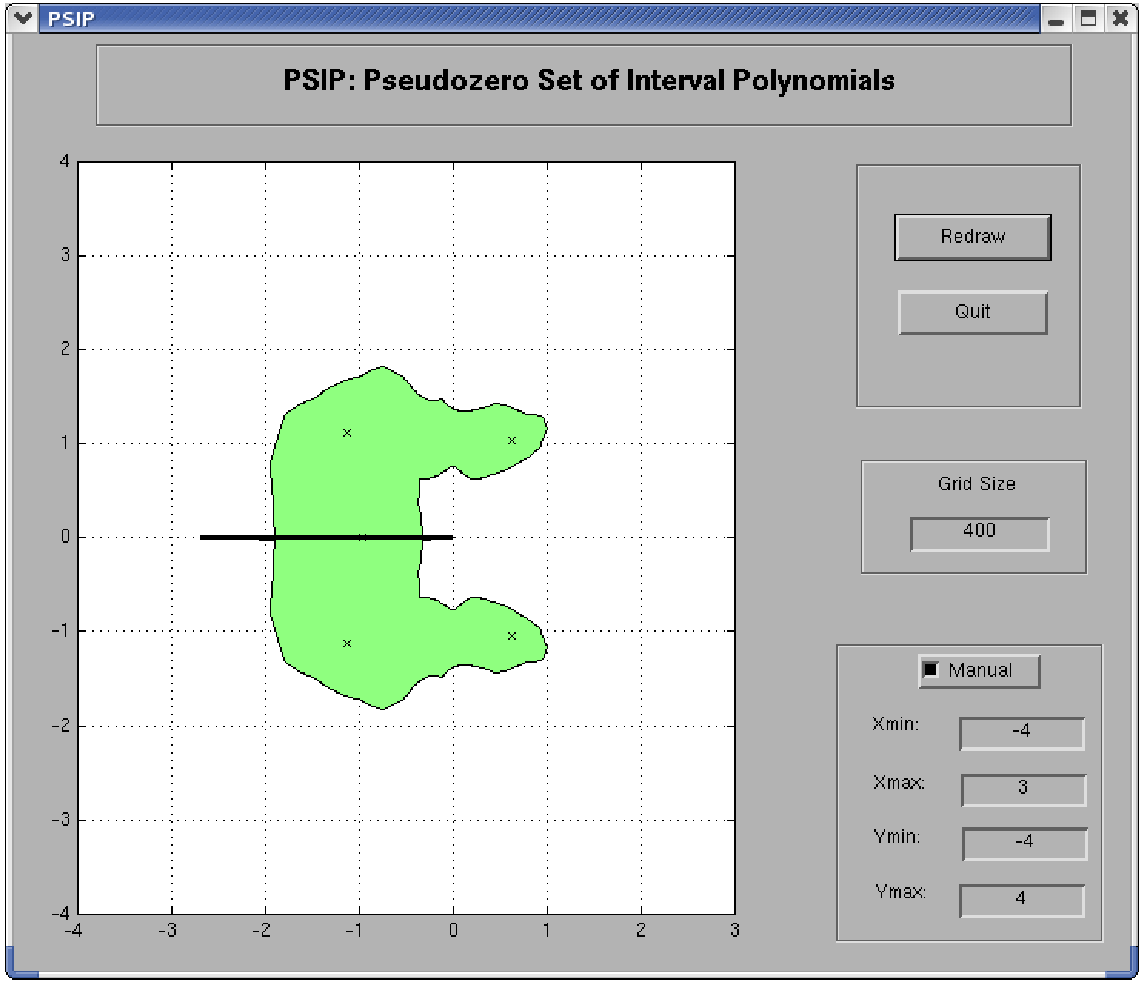
A graphical tool

A tool to draw zeros of interval polynomials



Pseudozeros : brief survey of existing references

- Mosier (1986) : Definition and study form the ∞-norm.
- Hinrichsen and Kelb : *spectral value sets*
- Trefethen and Toh (1994) : Study for the 2-norm.

  pseudozeros $\approx$ pseudospectra of the companion matrix.
- Chatelin and Frayssé (1996) : propose a Synthesis in *Lectures on Finite Precision Computations* (SIAM)
- Stetter (1999,2004) : *Numerical polynomial algebra*. Generalization of the previous works.
- Karow (2003) : thesis on *Spectral value sets*

$\Longrightarrow$ What about computing pseudozero set in finite precision ?

Validation of pseudozero set

Set Inversion via Interval Analysis

Set inversion problem

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R} \quad \text{and} \quad Y \subset \mathbb{R}$$

One wants to compute an inner and outer approximation of $X = f^{-1}(Y)$

For pseudozero sets, $Z_\varepsilon(p) = f^{-1}(Y)$ with

$$f(x, y) = \frac{|p(x+iy)|}{\|\underline{x+iy}\|_*}$$

and $Y = [0, \varepsilon]$

SIVIA algorithm (Jaulin,Walter,1993)

Inputs : inclusion function F of f, Y, feasible box $x(0)$, accuracy of the paving ε_r

Initialization : $k = 0$, stack $= \varnothing$, $K_{\mathrm{in}} = \varnothing$, $K_{\mathrm{i}} = \varnothing$

Iteration k

Step 1 : if $F(x(k)) \subset Y$, then $K_{\mathrm{in}} = K_{\mathrm{in}} \cup x(k)$. Go to step 4

Step 2 : if $F(x(k)) \cap Y = \varnothing$ then go to Step 4

Step 3 : if $w(x(k)) \leqslant \varepsilon_r$ then $K_{\mathrm{i}} = K_{\mathrm{i}} \cup x(k)$ else bisect $x(k)$ and stack

Step 4 : if stack is not empty, then unstack $x(k+1)$, increment k and goto Step 1

End

We have

$$K_{\mathrm{in}} \subset X \subset K_{\mathrm{out}} := K_{\mathrm{in}} \cup K_{\mathrm{i}}$$

Finite precision computation

Floating point operations in IEEE 754, $a, b \in \mathbb{F}$

$$\mathrm{fl}(a \circ b) = (a \circ b)(1+\varepsilon) \text{ for } \circ = \{+, -, \cdot, /\} \text{ and } |\varepsilon| \leqslant \text{eps}.$$

So that

$$|a \circ b - \mathrm{fl}(a \circ b)| \leqslant \text{eps}|a \circ b| \text{ and}$$
$$|a \circ b - \mathrm{fl}(a \circ b)| \leqslant \text{eps}|\,\mathrm{fl}(a \circ b)| \text{ for } \circ = \{+, -, \cdot, /\}.$$

For double precision, eps $= 2^{-53}$

We assume neither overflow nor underflow

Finite precision with polynomials

Evaluation of a real polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$, with $a_i, x \in \mathbb{F}$,

$$|p(x) - \mathrm{fl}(p(x))| \leqslant \gamma_{2n} \sum_{i=0}^{n} |a_i||x|^i = \gamma_{2n} \tilde{p}(|x|)$$

If $a_i \geqslant 0$ and $x \geqslant 0$ then

$$0 \leqslant p(x) \leqslant (1 + \mathsf{eps})^{2n} \, \mathrm{fl}(p(x))$$

Moreover, for $x \in \mathbb{F}$, [Ogita,Rump,Oishi,05]

$$(1 + \mathsf{eps})^n |x| \leqslant \frac{|x|}{(1 - \mathsf{eps})^n} \leqslant \frac{|x|}{1 - n\mathsf{eps}} |x| \leqslant \mathrm{fl}\left(\frac{|x|}{1 - (n+1)\mathsf{eps}} \right)$$

What is validation in finite precision ?

General form for the pseudozero set

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{q(z)} \leqslant \varepsilon \right\},$$

with $q(z) > 0$ for all $z \in \mathbb{C}$

Aim : find $\alpha(z) \in \mathbb{F}$ and $\beta(z) \in \mathbb{F}$ such that

$$\boxed{\alpha(z) \leqslant \frac{|p(z)|}{q(z)} \leqslant \beta(z)}$$

So that

$$\boxed{\begin{array}{lcl} \beta(z) \leqslant \varepsilon & \Rightarrow & z \in Z_\varepsilon(p) \\ \varepsilon \leqslant \alpha(z) & \Rightarrow & z \notin Z_\varepsilon(p) \end{array}}$$

Example with real polynomial with real zeros (1/3)

For real polynomial p with real zeros and real perturbations with ∞-norm,

$$Z_\varepsilon(p) = \left\{ x \in \mathbb{R} : |g(x)| := \frac{|p(x)|}{\sum_{i=0}^{n} |x|^j} \leqslant \varepsilon \right\},$$

2 ways :

- interval arithmetic
  Need directed rounding modes
- rigorous error bound in floating point arithmetic
  Use only rounded to nearest mode

Example with real polynomial with real zeros (2/3)

We have

$$\alpha(x) \leqslant \frac{|p(x)|}{q(x)} \leqslant \beta(x)$$

with

$$\alpha(x) = \mathrm{fl}\left(\xi_{2n+3} \cdot \left[|p(x)| - \gamma_{2n}\tilde{p}(|x|)/\xi_{2n+3}\right]/q(x)\right)$$

and

$$\beta(x) = \mathrm{fl}\left(\frac{\left[|p(x)| + \gamma_{2n}\tilde{p}(|x|)/\xi_{2n+3}\right]/q(x)}{\xi_{2n+3}}\right)$$

where $\xi_n = 1 - n\mathrm{eps} \in \mathbb{F}$.

Example with real polynomial with real zeros (3/3)

if the error is too big $\longrightarrow$ Compensated Horner Scheme[1]

Results are as accurate as if computed in twice the working precision

[1] S.Graillat, N.Louvet, Ph.Langlois. Compensated Horner Scheme. Submitted

Conclusion and future work

We have presented

- an algorithm to draw an inner and outer approximation of a pseudozero set
- a formula to test wether a point is inside or outside the pseudozero set for real polynomials

Future work

- a similar analysis for complex polynomials
- a Compensated Horner Scheme for complex polynomials