Pseudozero Set of Interval Polynomials

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Outline of the talk

I — Pseudozero set

- Definition and computation

II — Pseudozero set of interval polynomials

- Real pseudozero set of polynomials
- Presentation of PSIP

Pseudozeros : definition, computation and motivation

Pseudozero set : definition

Perturbation :

Neighborhood of polynomial p

$$N_\varepsilon(p) = \{\widehat{p} \in \mathbf{C}_n[z] : \|p - \widehat{p}\| \leqslant \varepsilon\}\,.$$

Definition of the ε-pseudozero set :

$$Z_\varepsilon(p) = \{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\}\,.$$

$\|\cdot\|$ a norm on the vector of the coefficients of p

This set is formed by the zeros of polynomials "near p".

Pseudozeros : brief survey of existing references

- Mosier (1986) : Definition and study form the ∞-norm.
- Hinrichsen and Kelb : *spectral value sets*
- Trefethen and Toh (1994) : Study for the 2-norm.

  pseudozeros $\approx$ pseudospectra of the companion matrix.
- Chatelin and Frayssé (1996) : propose a Synthesis in *Lectures on Finite Precision Computations* (SIAM)
- Stetter (1999,2004) : *Numerical polynomial algebra*. Generalization of the previous works.
- Zhang (2001) : Study of the influence of the basis for the 2-norm (condition number of the evaluation).
- Karow (2003) : thesis on *Spectral value sets*

Pseudozeros are easily computable

Theorem :
The ε-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|\underline{z}\|_*} \leqslant \varepsilon \right\},$$

where $\underline{z} = (1, z, \dots, z^n)$ and $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$,

$$\|y\|_* = \sup_{x \neq 0} \frac{|y^* x|}{\|x\|}$$

Algorithm of computation

Algorithm to draw the ε-pseudozero set :

1. We mesh a square containing all the roots of p (MATLAB command : `meshgrid`).
2. We compute $g(z) := \frac{|p(z)|}{\|\underline{z}\|_*}$ for all the nodes z in the grid.
3. We draw the contour level $|g(z)| = \varepsilon$ (MATLAB command : `contour`).

A famous example

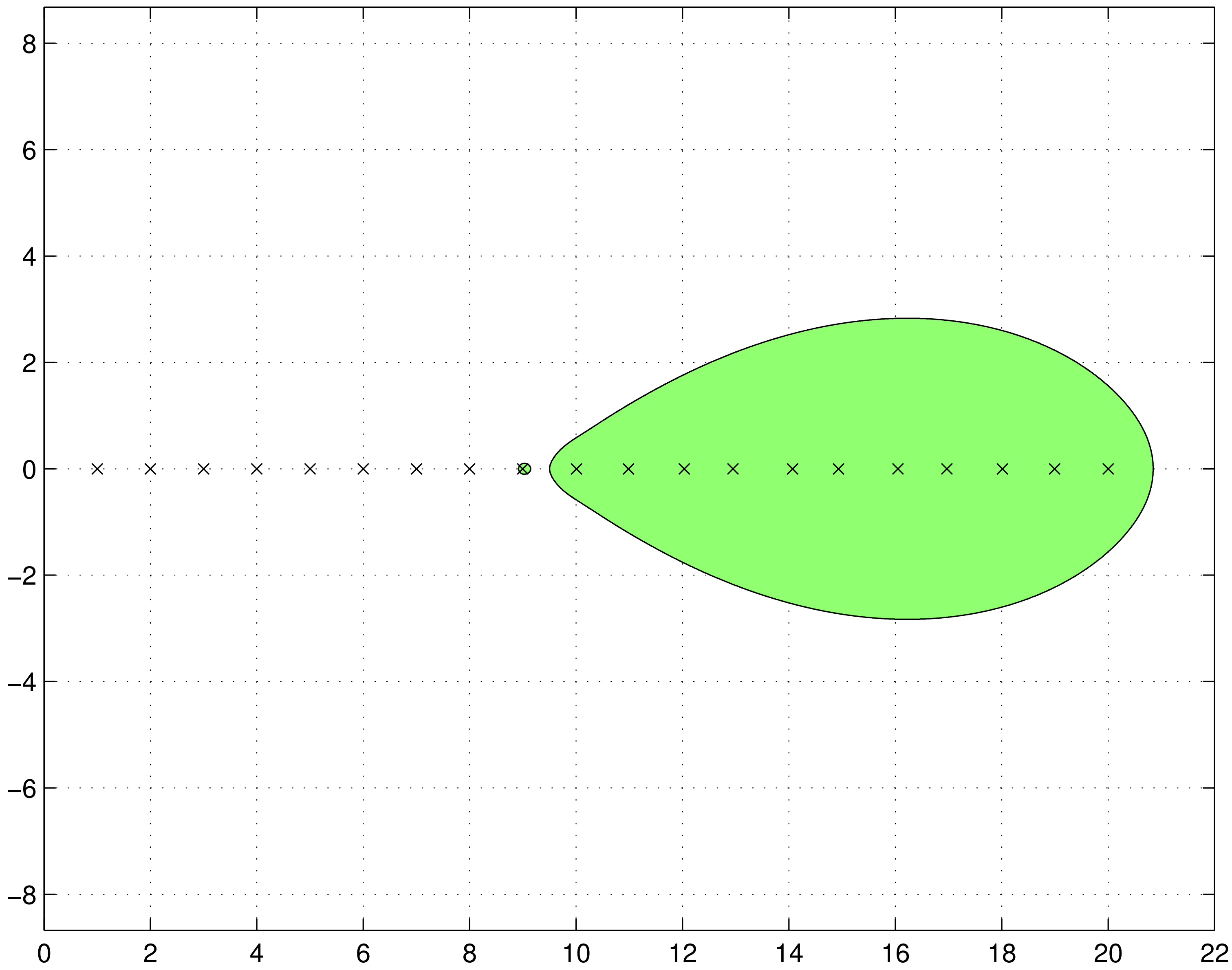
Pseudozero set of the *Wilkinson* polynomial

$$\begin{aligned} W_{20} &= (z-1)(z-2)\cdots(z-20), \\ &= z^{20} - 210z^{19} + \cdots + 20!. \end{aligned}$$

We perturb only the coefficient of z^{19} with $\varepsilon = 2^{-23}$.
One use the weighted-norm $\|\cdot\|_\infty$:

$$\|p\|_\infty = \max_i \frac{|p_i|}{m_i} \text{ with } m_i \text{ non negative}$$

with $m_{19} = 1$, $m_i = 0$ otherwise and the convention $m/0 = \infty$ if $m > 0$ and $0/0 = 0$.



8
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12
14
16
18
20
22

Pseudozero set of interval polynomials

Interval polynomial

An interval polynomial : polynomial whose coefficients are real intervals.

We denote by $\mathbf{IR}[z]$ the set of interval polynomials and by $\mathbf{IR}_n[z]$ the set of interval polynomials with degree at most n.

Let $p \in \mathbf{IR}_n[z]$. We can write

$$p(z) = \sum_{i=0}^{n} [a_i, b_i] z^i.$$

The zeros of an interval polynomial is the set

$$\mathbf{Z}(p) := \{z \in \mathbf{C} : \text{there exist } m_i \in [a_i, b_i], i = 0 : n \text{ such that} \sum_{i=0}^{n} m_i z^i = 0\}.$$

$\Longrightarrow$ Compute $\mathbf{Z}(p)$.

Definition of real pseudozero set

Let $p = \sum_{i=0}^{n} p_i z^i$ be a polynomial of $\mathbf{R}_n[z]$

Perturbations :

Real neighborhood of p

$$N_\varepsilon^R(p) = \{\widehat{p} \in \mathbf{R}_n[z] : \|p - \widehat{p}\| \leqslant \varepsilon\}.$$

Definition of the real ε-pseudozero set

$$Z_\varepsilon^R(p) = \left\{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon^R(p)\right\}.$$

Computation of the real pseudozero set

Theorem :

The real ε-pseudozero set satisfies

$$Z_\varepsilon^R(p) = Z(p) \cup \left\{ z \in \mathbf{C} \backslash Z(p) : h(z) := d(G_R(z), \mathbf{R} G_I(z)) \geqslant \frac{1}{\varepsilon} \right\},$$

where d is defined for $x, y \in \mathbf{R}^{n+1}$ by

$$d(x, \mathbf{R}y) = \inf_{\alpha \in \mathbf{R}} \|x - \alpha y\|_*$$

and where $G_R(z)$, $G_I(z)$ are the real and imaginary part of

$$G(z) = \frac{1}{p(z)} (1, z, \ldots, z^n)^T, \; z \in \mathbf{C} \backslash Z(p)$$

Can be viewed as a special case of *spectral value set* [Karow 03]

What for $\mathbf{R} \cap Z_\varepsilon^R(p)$?

Lemma. *Given $z \in \mathbf{R}$, z belongs to $Z_\varepsilon^R(p)$ if and only if z belongs to $Z_\varepsilon(p)$.*

Draw the complex pseudozero set or the real pseudozero set on the real axis is similar.

Some properties

The function d defined for $x, y \in \mathbf{R}^{n+1}$ by

$$d(x, \mathbf{R}y) = \inf_{\alpha \in \mathbf{R}} \|x - \alpha y\|_*$$

satisfies

$$d(x, \mathbf{R}y) = \begin{cases} \sqrt{\|x\|_2^2 - \frac{\langle x, y \rangle^2}{\|y\|_2^2}} & \text{if } y \neq 0, \\ \|x\|_2 & \text{if } y = 0 \end{cases} \qquad \text{for the norm} \|\cdot\|_2$$

$$d(x, \mathbf{R}y) = \begin{cases} \min_{\substack{i=0:n \\ y_i \neq 0}} \|x - (x_i/y_i)y\|_1 & \text{if } y \neq 0, \\ \|x\|_1 & \text{if } y = 0 \end{cases} \qquad \text{for the norm } \|\cdot\|_\infty$$

Some properties (cont'd)

Proposition 1 :
The real ε-pseudozero set $Z_\varepsilon^R(p)$ is symmetric with respect to the real axis.

Proposition 2 :
The real ε-pseudozero set $Z_\varepsilon^R(p)$ is included in the complex ε-pseudozero set.

Algorithm to draw real pseudozero set

Drawing of real ε-pseudozero set :

1. We mesh a square containing all the roots of p (MATLAB command : `meshgrid`).
2. We compute $h(z) := d(G_R(z), \mathbf{R}G_I(z))$ for all the nodes z in the grid.
3. We draw the contour level $|h(z)| = \frac{1}{\varepsilon}$ (MATLAB command : `contour`).

Pseudozero set with weighted norm

$$p(z) = \sum_{i=0}^{n} p_i z^i.$$

- identification of p with the vector $(p_0, p_1, \dots, p_n)^T$
- $d := (d_0, \dots, d_n)^T \in \mathbf{R}^{n+1}$ represents the weight of the coefficients of p
- $\| \cdot \|_{\infty,d}$ defined by

$$\|p\|_{\infty,d} = \max_{i=0:n} \{|p_i|/|d_i|\}.$$

Zeros of interval polynomials and real pseudozero set

Let us denote p_c the central polynomial defined by

$$p_c(z) = \sum_{i=0}^{n} c_i z^i,$$

with $c_i = (a_i + b_i)/2$.

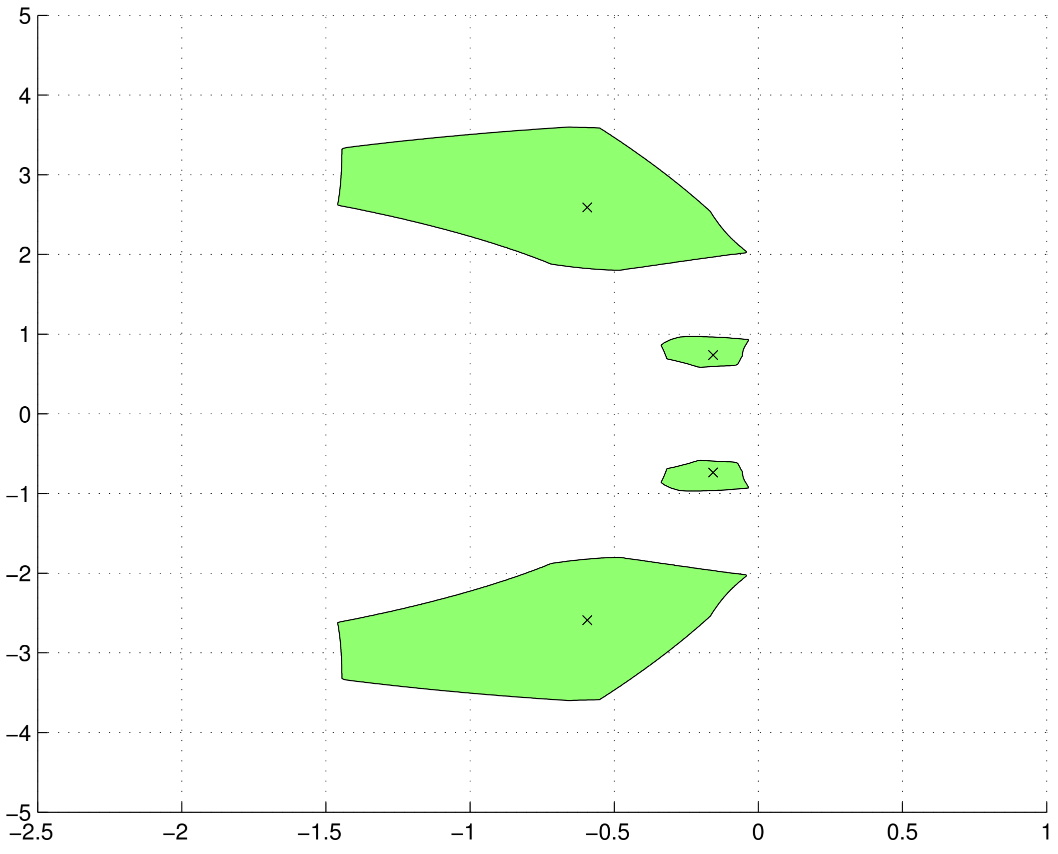
Let us denote $d_i := (b_i - a_i)/2$.

Proposition :
With the notation above, we have

$$\mathbf{Z}(p) = Z_\varepsilon^R(p_c) \text{ with } \varepsilon = 1.$$

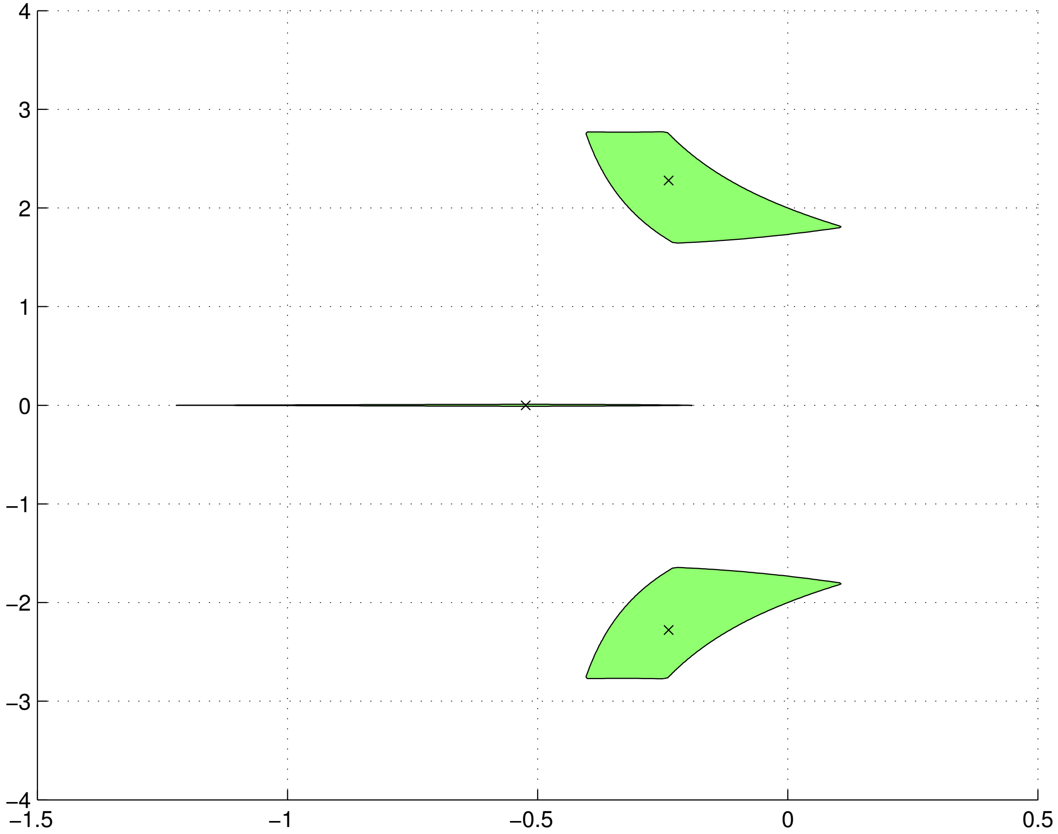
Example 1

$$p(z) = [1, 2]z^4 + [3, 3.2]z^3 + [10, 14]z^2 + [3, 5\sqrt{2}]z + [5, 7]$$



Example 2

$$p(z) = z^3 + z^2 + [3, 8]z + [1.5, 4]$$



Problem : choice of the grid

Lemma :
Let $p(z) = \sum_{i=0}^{n} [a_i, b_i] z^i$ an interval polynomial and

$$R := 1 + \frac{\max_{i=0:n}\{\max\{|a_i|, |b_i|\}\}}{\min\{|a_n|, |b_n|\}}.$$

Then

$$\mathbf{Z}(p) \subset B(O, R),$$

where $B(O, R)$ the ball in $\mathbf{C}$ of centre O and radius R.

Problems : discontinuities

Lemma [Hinrichsen et Kelb] :
The function

$$d : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}_+, \quad (x, y) \mapsto d(x, \mathbf{R}y)$$

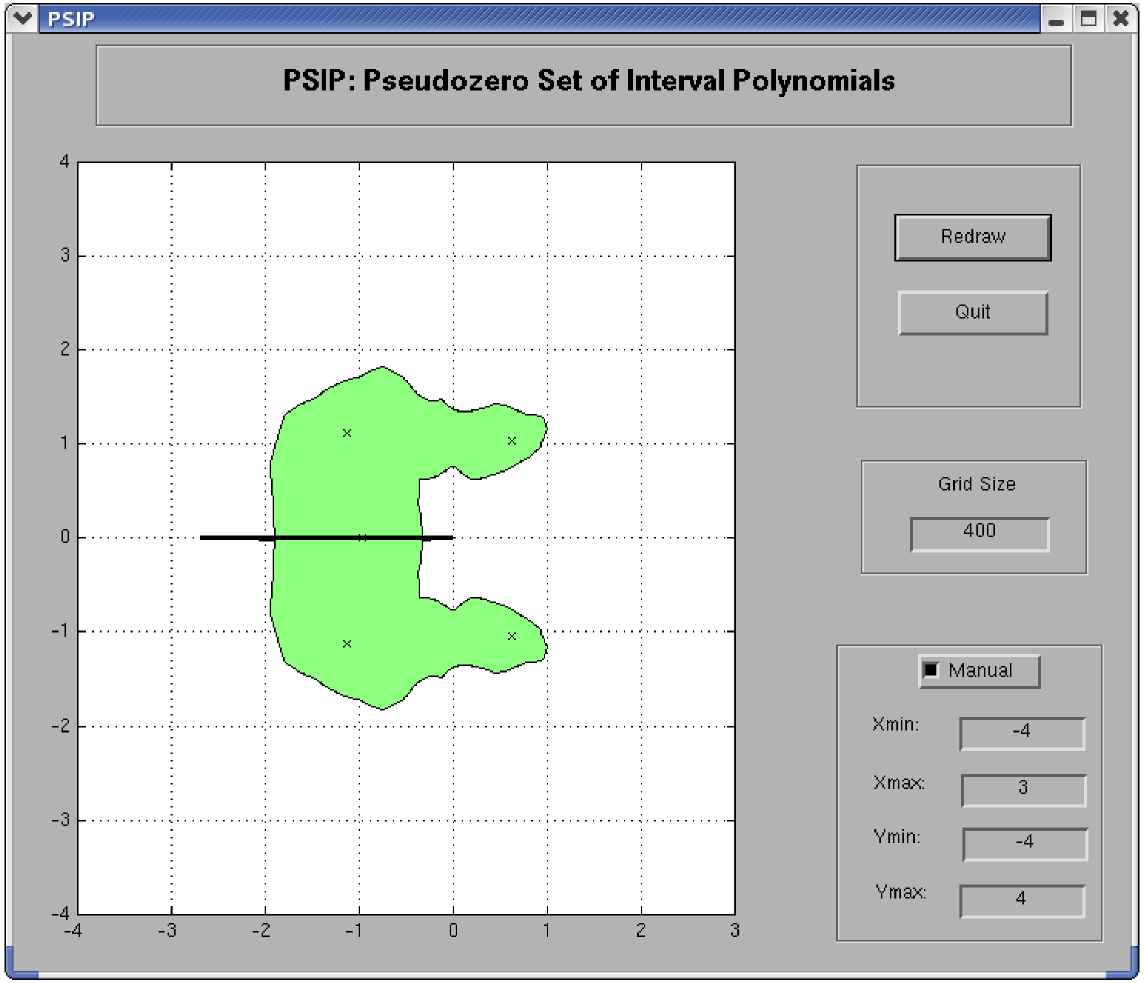
is continue for all (x, y) with $y \neq 0$ or $x = 0$ and discontinue for all $(x, 0) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, $x \neq 0$.

$\Longrightarrow$ Those discontinuties imply some difficulties for drawing near the real axis.

Solution : on the real axis, we draw complex pseudozero set.

Presentation of PSIP

A tool to draw zeros of interval polynomials



Presentation of PSIP (cont'd)

- a graphical interface for MATLAB (version 6.5 (R13))
- computation of grid that contains all the zeros
- possibilities of zoom and mesh refinement

Limitations :

- problem if the leading interval contains 0
- problems with discontinuities

Conclusion and future work

We have presented

- a tool to draw pseudozero set of interval polynomial

Future work

- certification of the drawing