Structured Perturbations in Scalar Product Spaces

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Motivations (1/2)

- Condition numbers and backward errors play an important role in numerical linear algebra.

$$\text{forward error} \lesssim \text{condition number} \times \text{backward error.}$$

- Growing interest in structured perturbation analysis.
- Substantial development of algorithms for structured problems.
- Backward error analysis of structure preserving algorithms may be difficult.

Motivations (2/2)

- For symmetric linear systems and for distances measured in the 2– or Frobenius norm:
  It makes no difference whether perturbations are restricted to be symmetric or not.
- Same holds for skew-symmetric and persymmetric structures. [S. Rump, 03].

Our contribution:

Extend and unify these results to

- Structured matrices in Lie and Jordan algebras,
- Several structured matrix problems.

Structured Problems

- Normwise structured condition numbers for
  - Linear systems,
  - Matrix inversion,
  - Nearness to singularity.
- Normwise structured backward errors for
  - Linear systems,
  - Eigenvalue problems.

Scalar Product

A scalar product $\langle\cdot,\cdot\rangle_{M}$ is a nondegenerate (M nonsingular) bilinear or sesquilinear form on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

$$\langle x, y\rangle_{M} = \begin{cases} x^T M y, & \text{real or complex bilinear forms,} \\ x^* M y, & \text{sesquilinear forms.} \end{cases}$$

Adjoint $A^\star$ of $A \in \mathbb{K}^{n\times n}$ wrt $\langle\cdot,\cdot\rangle_{M}$:

$$\langle A^\star x, y\rangle_{M} = \langle x, A^\star y\rangle_{M}, \ \forall x, y \in \mathbb{K}^n,$$

$$A^\star = \begin{cases} M^{-1} A^T M, & \text{for bilinear forms,} \\ M^{-1} A^* M, & \text{for sesquilinear forms.} \end{cases}$$

$\langle\cdot,\cdot\rangle_{M}$ *orthosymmetric* if $\begin{cases} M^T = \pm M, & (\textit{bilinear}), \\ M^* = \alpha M, |\alpha| = 1, & (\textit{sesquilinear}). \end{cases}$

Matrix Groups, Jordan and Lie Algebras

Three important classes of matrices associated with $\langle\cdot,\cdot\rangle_M$:

Automorphism group: $\mathbb{G} = \{A \in \mathbb{K}^{n\times n} : A^\star = A^{-1}\}$

Lie algebra: $\mathbb{L} = \{A \in \mathbb{K}^{n\times n} : A^\star = -A\}$.

Jordan algebra: $\mathbb{J} = \{A \in \mathbb{K}^{n\times n} : A^\star = A\}$.

Recall that

$$A^\star = \begin{cases} M^{-1}A^TM, & \text{for bilinear forms,} \\ M^{-1}A^*M, & \text{for sesquilinear forms.} \end{cases}$$

Concentrate on Jordan and Lie algebras of orthosymmetric scalar products $\langle\cdot,\cdot\rangle_M$ with M unitary.

Some Structured Matrices

Space	M	Jordan Algebra	Lie Algebra
Bilinear forms			
$\mathbb{R}^n$	I	Symm.	Skew-symm.
$\mathbb{C}^n$	I	Complex symm.	Complex skew-symm.
$\mathbb{R}^n$	R	Persymmetric	Perskew-symm.
$\mathbb{R}^n$	$\Sigma_{p,q}$	Pseudo symm.	Pseudo skew-symm.
$\mathbb{R}^{2n}$	J	Skew-Hamiltonian.	Hamiltonian
Sesquilinear form			
$\mathbb{C}^n$	I	Hermitian	Skew-Herm.
$\mathbb{C}^n$	$\Sigma_{p,q}$	Pseudo Hermitian	Pseudo skew-Herm.
$\mathbb{C}^{2n}$	J	J-skew-Hermitian	J-Hermitian

$$R = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \qquad \Sigma_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

Key Tools

$$\text{Define } \begin{aligned} \mathrm{Sym}(\mathbb{K}) &= \{A \in \mathbb{K}^{n\times n} : A^T = A\}, \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}, \\ \mathrm{Skew}(\mathbb{K}) &= \{A \in \mathbb{K}^{n\times n} : A^T = -A\}, \\ \mathrm{Herm}(\mathbb{C}) &= \{A \in \mathbb{C}^{n\times n} : A^* = A\}. \end{aligned}$$

$\mathbb{S}$: Lie algebra $\mathbb{L}$ or Jordan algebra $\mathbb{J}$ of *orthosymm.* $\langle \cdot, \cdot \rangle_M$.

$$M \cdot \mathbb{S} = \begin{cases} \mathrm{Sym}(\mathbb{K}) & \text{if } \begin{cases} M = M^T \text{ and } \mathbb{S} = \mathbb{J}, \\ M = -M^T \text{ and } \mathbb{S} = \mathbb{L}, \end{cases} \\ \mathrm{Skew}(\mathbb{K}) & \text{if } \begin{cases} M = M^T \text{ and } \mathbb{S} = \mathbb{L}, \\ M = -M^T \text{ and } \mathbb{S} = \mathbb{J}. \end{cases} \end{cases} \quad \text{(bilinear forms)}$$

$$M \cdot \mathbb{S} = \begin{cases} \mathrm{Herm}(\mathbb{C}) & \text{if } \mathbb{S} = \mathbb{J}, \\ i\, \mathrm{Herm}(\mathbb{C}) & \text{if } \mathbb{S} = \mathbb{L}. \end{cases} \quad \text{(sesquilinear forms)}$$

Linear Systems

Structured condition number for linear system $Ax = b$, $x \neq 0$:

$$\mathrm{cond}_\nu(A, x; \mathbb{S}) = \lim_{\varepsilon\to 0} \sup \left\{ \frac{\|\Delta x\|_2}{\varepsilon \|x\|_2} : (A + \Delta A)(x + \Delta x) = b + \Delta b, \frac{\|\Delta A\|_\nu}{\|A\|_\nu} \leq \varepsilon, \frac{\|\Delta b\|_2}{\|b\|_2} \leq \varepsilon, A + \Delta A \in \mathbb{S} \right\}, \quad \nu = 2, F.$$

$\mathbb{S}$: Jordan or Lie algebra of $\langle \cdot, \cdot \rangle_M$ orthosymm. with M unitary.

For nonsingular $A \in \mathbb{S}$, $x \neq 0$ and $\nu = 2, F$,

$$\frac{\mathrm{cond}_\nu(A, x; \mathbb{C}^{n\times n})}{\sqrt{2}} \leq \mathrm{cond}_\nu(A, x; \mathbb{S}) \leq \mathrm{cond}_\nu(A, x; \mathbb{C}^{n\times n}).$$

Linear Systems with Nonlinear Structures

Structured condition number for linear system $Ax = b$, $x \neq 0$:

$$\mathrm{cond}_\nu(A, x; \mathbb{G}) = \lim_{\varepsilon \to 0} \sup \left\{ \frac{\|\Delta x\|_2}{\varepsilon \|x\|_2} : (A + \Delta A)(x + \Delta x) = b + \Delta b, \frac{\|\Delta A\|_\nu}{\|A\|_\nu} \leq \varepsilon, \frac{\|\Delta b\|_2}{\|b\|_2} \leq \varepsilon, A + \Delta A \in \mathbb{G} \right\}, \; \nu = 2, F.$$

$\mathbb{G}$: automorphism group of $\langle \cdot, \cdot \rangle_M$ orthosymm. with M unitary.

For nonsingular $A \in \mathbb{G}$, $x \neq 0$ and $\nu = 2, F$,

$$\gamma \frac{\mathrm{cond}_\nu(A, x; \mathbb{C}^{n \times n})}{\|A\|_2 \|A^{-1}\|_2} \leq \mathrm{cond}_\nu(A, x; \mathbb{G}) \leq \mathrm{cond}_\nu(A, x; \mathbb{C}^{n \times n}).$$

where $\gamma = 1/\sqrt{2}$ if $\nu = 2$ and $\gamma = 1/2$ if $\nu = F$.

Matrix Inversion

Structured condition number for matrix inverse ($\nu = 2, F$):

$$\kappa_\nu(A;\mathbb{S}) := \lim_{\varepsilon\to 0}\sup\left\{\frac{\|(A+\Delta A)^{-1} - A^{-1}\|_\nu}{\varepsilon\|A^{-1}\|_\nu} : \frac{\|\Delta A\|_\nu}{\|A\|_\nu} \le \varepsilon, A+\Delta A \in \mathbb{S}\right\}.$$

$\mathbb{S}$: Jordan or Lie algebra of orthosymm. $\langle\cdot,\cdot\rangle_M$ with M unitary.

For nonsingular $A \in \mathbb{S}$,

$$\begin{aligned}\kappa_2(A;\mathbb{S}) &= \kappa_2(A;\mathbb{C}^{n\times n}) = \|A\|_2\|A^{-1}\|_2,\\ \kappa_F(A;\mathbb{S}) &= \kappa_F(A;\mathbb{C}^{n\times n}) = \frac{\|A\|_F\|A^{-1}\|_2^2}{\|A^{-1}\|_F}.\end{aligned}$$

Matrix Inversion with Nonlinear Structures

$\mathbb{G}$: automorphism group of orthosymm. $\langle \cdot, \cdot \rangle_M$ with M unitary.

B a pattern matrix for $T_A\mathbb{G} = A \cdot \mathbb{L}$, *i.e.*,
for every $E \in T_A\mathbb{G}$ there exists a uniquely defined parameter vector p with

$$\mathrm{vec}(E) = Bp, \quad \|E\|_F = \|p\|_2.$$

For nonsingular $A \in \mathbb{G}$,

$$\kappa_F(A; \mathbb{G}) = \frac{\|A\|_F}{\|A^{-1}\|_F} \|(A^T \otimes A)^{-1} B\|_2$$

Distance to Singularity

Structured distance to singularity ($\nu = 2, F$):

$$\delta_\nu(A;\mathbb{S}) = \min\left\{\varepsilon : \frac{\|\Delta A\|_\nu}{\|A\|_\nu} \le \varepsilon, A + \Delta A \text{ singular}, \Delta A \in \mathbb{S}\right\}.$$

$\mathbb{S}$: Jordan or Lie algebra of $\langle\cdot,\cdot\rangle_M$ orthosymm. with M unitary.

For nonsingular $A \in \mathbb{S}$,

$$\begin{aligned} \delta_2(A;\mathbb{S}) &= \delta_2(A;\mathbb{C}^{n\times n}) = \frac{1}{\|A\|_2\|A^{-1}\|_2}, \\ \delta_F(A;\mathbb{C}^{n\times n}) \le \delta_F(A;\mathbb{S}) &\le \sqrt{2}\,\delta_F(A;\mathbb{C}^{n\times n}). \end{aligned}$$

Structured Backward Errors (1/2)

Structured backward error ($\nu = 2, F$):

$$\mu_\nu(y, r, \mathbb{S}) = \min\{\|\Delta A\|_\nu : \ \Delta A y = r, \ \Delta A \in \mathbb{S}\}.$$

- For linear systems: $y \neq 0$ is the approx. sol. to $Ax = b$ and $r = b - Ay$.
- For eigenproblems: (y, λ) approx. eigenpair of A, $r = (\lambda I - A)y$.

$\mathbb{S}$: Jordan or Lie algebra of $\langle\cdot,\cdot\rangle_M$ orthosymm. with M unitary.

$\mu_\nu(y, r, \mathbb{S}) \neq \infty$ iff y, r satisfies the conditions:

$M \cdot \mathbb{S}$	Condition
$\mathrm{Sym}(\mathbb{K})$	none
$\mathrm{Skew}(\mathbb{K})$	$r^T y = 0$
$\mathrm{Herm}(\mathbb{C})$	$r^* y \in \mathbb{R}$.

Structured Backward Errors (2/2)

$$\mu_\nu(y, r, \mathbb{S}) = \min\{\|\Delta A\|_\nu : \ \Delta A y = r, \ \Delta A \in \mathbb{S}\}, \quad \nu = 2, F.$$

Recall $\mu_\nu(y, r; \mathbb{C}^{n\times n}) = \|r\|_2 / \|y\|_2$.

$\mathbb{S}$: Jordan or Lie algebra of $\langle\cdot,\cdot\rangle_M$ orthosymm. with M unitary.
If $\mu_\nu(y, r, \mathbb{S}) \neq \infty$ ($\nu = 2, F$),

$$\mu_\nu(y, r; \mathbb{C}^{n\times n}) \leq \mu_\nu(y, r; \mathbb{S}) \leq \sqrt{2}\, \mu_\nu(y, r; \mathbb{C}^{n\times n}).$$

In particular for $\nu = F$,

$$\mu_F(y, r; \mathbb{S}) = \frac{1}{\|y\|_2} \sqrt{2\|r\|_2^2 - \frac{|\langle y, r\rangle_M|^2}{\beta^2 \|y\|_2^2}}.$$

Conclusion

For matrices in Jordan or Lie algebra of $\langle \cdot, \cdot \rangle_M$ orthosymm. with M unitary,

- Usual unstructured perturbation analysis sufficient for
  - linear system,
  - matrix inversion,
  - distance to singularity.
- Structured backward error:
  - may be ∞,
  - when finite, is within a small factor of the unstructured one.
- Eigenvalue condition number:
  - Recent results from Karow, Kressner and Tisseur (2005).

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