Pseudozero set decides on polynomial stability

Stef GRAILLAT
Université de Perpignan
graillat@univ-perp.fr
http://gala.univ-perp.fr/~graillat
Joint work with Philippe LANGLOIS

Sixteenth International Symposium on
Mathematical Theory of Networks and Systems

Katholieke Universiteit Leuven, Belgium, July 5-9, 2004

Motivations

Polynomial coefficients are often approximate values

Three well known sources of approximation are considered in scientific computation :

(1) errors due to discretization and truncation,

(2) errors due to roundoff, and

(3) errors due to uncertainty in the data.

$\Longrightarrow$ Use tools designed for such approximate polynomials in control theory

Outline of the talk

1 — Pseudozero set

- Definition
- Computation

2 — Applications of pseudozeros in control theory

- Robust stability for polynomials
- Stability radius for polynomials

Pseudozero set : definition

Let p be a given polynomial of $\mathbf{C}_n[z]$

Perturbation :

Neighborhood of polynomial p

$$N_\varepsilon(p) = \{\widehat{p} \in \mathbf{C}_n[z] : \|p - \widehat{p}\| \leqslant \varepsilon\}\,.$$

Definition of the ε-pseudozero set :

$$Z_\varepsilon(p) = \{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\}\,.$$

$\|\cdot\|$ a norm on the vector of the coefficients of p

Pseudozero set : the set of the zeros of polynomials "near p".

Pseudozeros are easily computable

Theorem [Stetter] :
The ε-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|\underline{z}\|_*} \leqslant \varepsilon \right\},$$

where $\underline{z} = (1, z, \ldots, z^n)$ and $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$,

$$\|y\|_* = \sup_{x \neq 0} \frac{|y^* x|}{\|x\|}$$

Pseudozero set : algorithm of computation

1. We mesh a square containing all the roots of p (MATLAB command : `meshgrid`).
2. We compute $g(z) := \frac{|p(z)|}{\|\underline{z}\|_*}$ for all the nodes z of the grid.
3. We plot the contour level $|g(z)| = \varepsilon$ (MATLAB command : `contour`).

Initialization :

- Find a square containing all the roots of p and all the pseudozeros.
- Find a grid step that separates all the roots.

A famous example

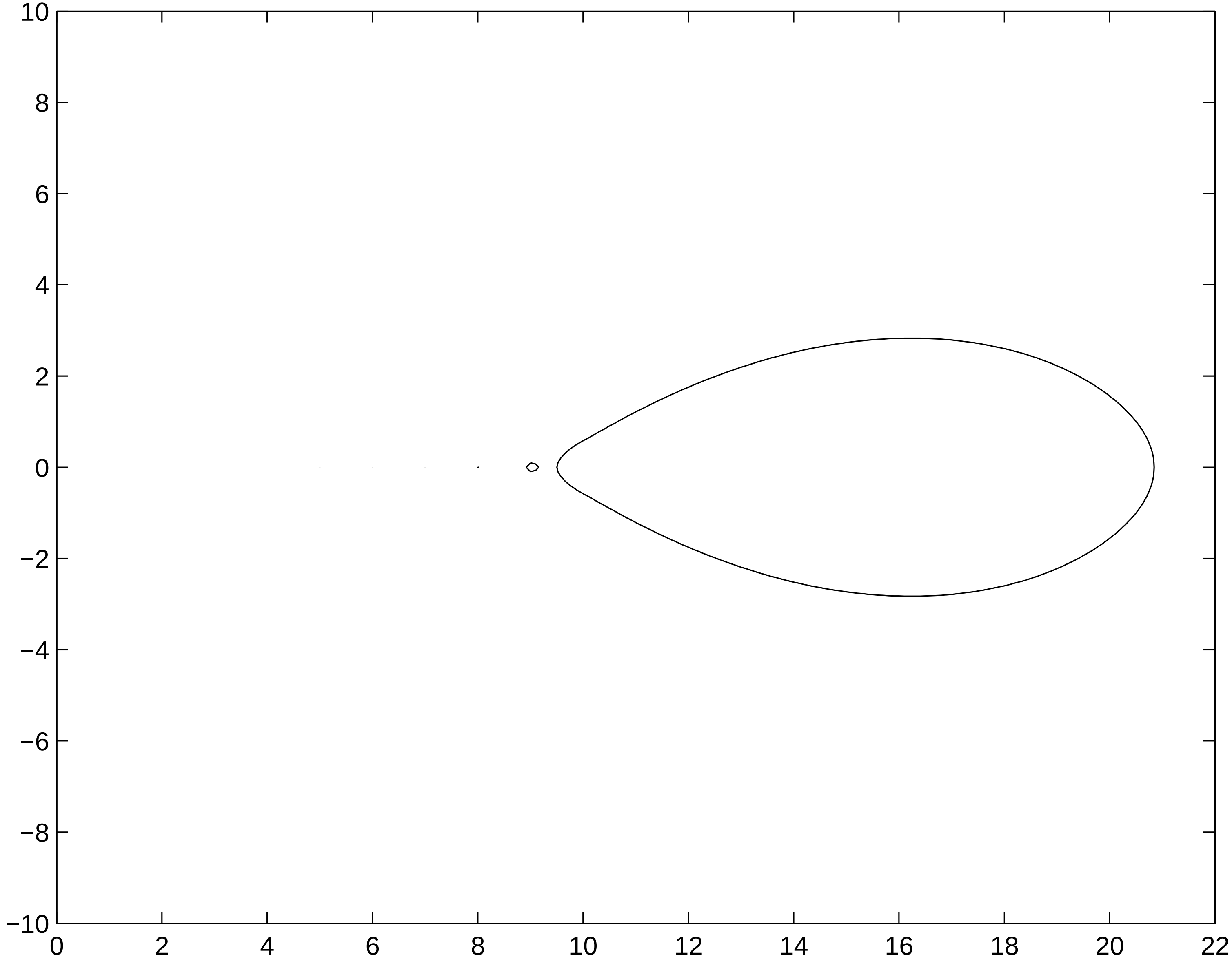
Pseudozero set of the *Wilkinson* polynomial

$$\begin{aligned} W_{20} &= (z-1)(z-2)\cdots(z-20), \\ &= z^{20} - 210z^{19} + \cdots + 20!. \end{aligned}$$

We only perturb the coefficient of z^{19} with $\varepsilon = 2^{-23}$.
One uses the weighted-norm $\|\cdot\|_\infty$:

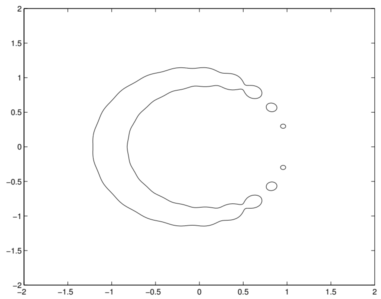
$$\|p\|_\infty = \max_i \frac{|p_i|}{m_i} \text{ with } m_i \text{ non negative}$$

with $m_{19} = 1$, $m_i = 0$ otherwise and the convention $m/0 = \infty$ if $m > 0$ and $0/0 = 0$.

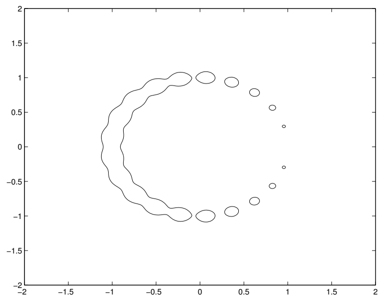

10
8
6
4
2
0
−2
−4
−6
−8
−10
0
2
4
6
8
10
12
14
16
18
20
22

Evolution of ε-pseudozero w.r.t ε

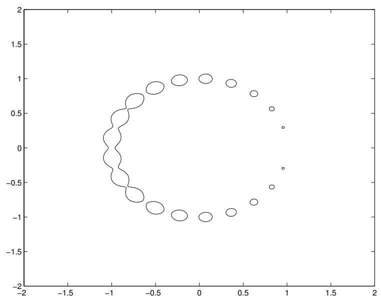
Pseudozero set of the polynomial $p(z) = 1 + z + \cdots + z^{20}$ for different values of ε.



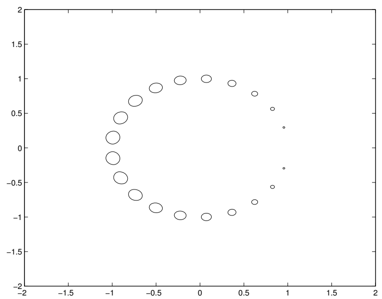
(a) $\varepsilon = 10^{-1}$



(b) $\varepsilon = 10^{-1.2}$



(c) $\varepsilon = 10^{-1.3}$



(d) $\varepsilon = 10^{-1.4}$

Pseudozeros : brief survey of existing references

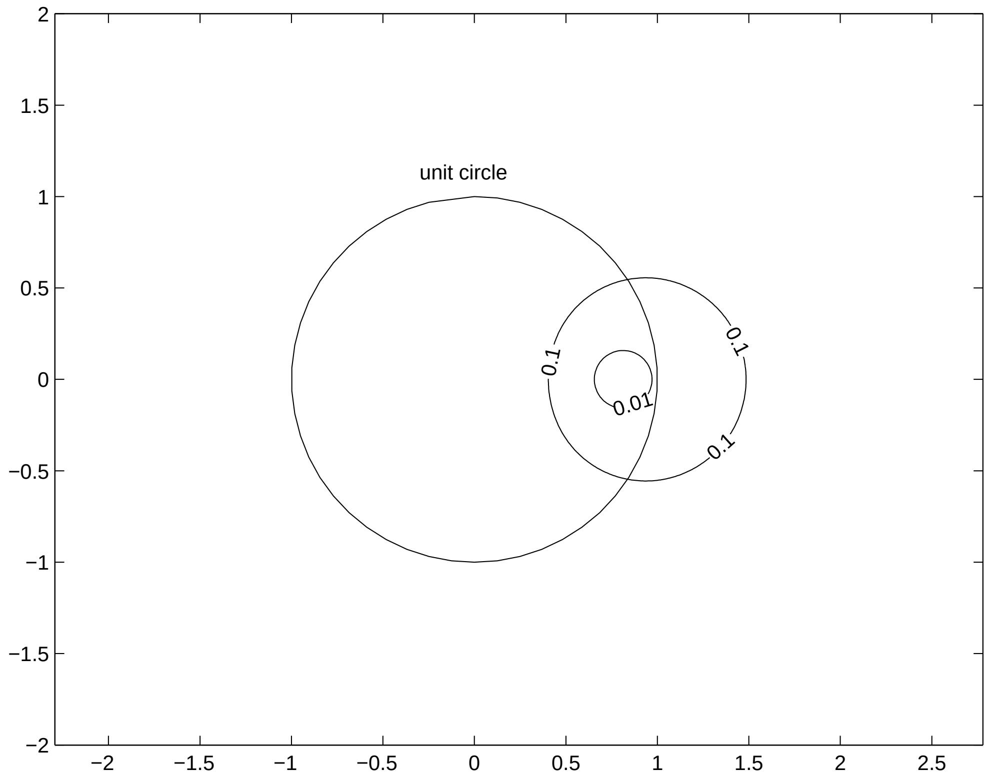
- Mosier (1986) : Definition and study for the ∞-norm.
- Hinrichsen and Kelb (1993) : Spectral value sets.
- Trefethen and Toh (1994) : Study for the 2-norm.
  pseudozeros $\approx$ pseudospectra of the companion matrix.
- Zhang (2001) : Study the influence of the basis for the 2-norm (condition number of the evaluation).
- Stetter (2004) : *Numerical Polynomial Algebra* (SIAM). Generalization of the previous works.

Other applications of pseudozero set : Robust stability and Stability radius for polynomials

Schur robust stability in control theory

Schur stability : $|\text{roots of } p| < 1$.

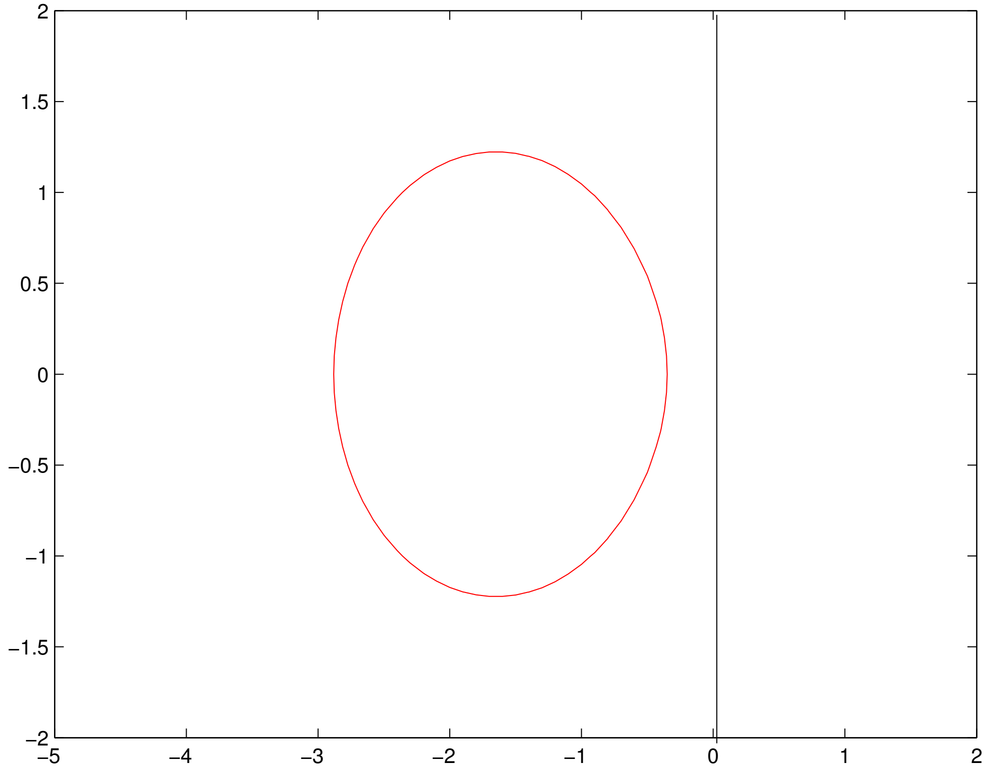
ε-pseudozero set of $p(z) = (z - 0.8)^2$ for $\varepsilon = 0.1$ and $\varepsilon = 0.01$.



Hurwitz robust stability in control theory

Hurwitz stability : Real part of roots of $p < 0$.

ε-pseudozero set of $p(z) = (z+1)^2$ for $\varepsilon = 0.4$.



Computation of stability radius

$\mathcal{P}_n$: polynomials of $\mathbf{C}[X]$ of degree at most n
$\mathcal{M}_n$: monic polynomials of $\mathcal{P}_n$ of degree n
$\|\cdot\|$: the 2-norm of the coefficients of a polynomial

Definition. *A polynomial is* stable *if all its roots have negative real part and* unstable *otherwise (Hurwitz stability).*

The function *abscissa* $a : \mathcal{P} \to \mathbf{R}$ is defined by

$$a(p) = \max\{\mathrm{Re}(z) : p(z) = 0\}.$$

A polynomial p is stable $\Longleftrightarrow a(p) < 0$

Motivation

In control theory, transfer functions are often written as $H(p) = \frac{N(p)}{D(p)}$ where N and D are polynomials.

The system is stable if D is a stable polynomial .

Question : if D is stable, how far is it from an unstable system ?

Problem : Find the distance to the nearest unstable system.

(we assume that D is monic)

How to compute the stability radius

Stability radius $\beta(p)$: distance of the polynomial $p \in \mathcal{M}_n$ from the set of monic unstable polynomials.

$$\beta(p) = \min\{\|p - q\| : q \in \mathcal{M}_n \text{ and } a(q) \geqslant 0\}.$$

Statement of the problem :

Given a polynomial $p \in \mathcal{M}_n$, let us compute $\beta(p)$.

Our solution

Tools

- an explicit formula that defines the pseudozeros
- the continuous dependency of the roots w.r.t the polynomial coefficients
- Sturm sequences to count the real roots

The results

- an algorithm that approximates $\beta(p)$ up to an arbitrary accuracy τ
- a plot showing the pseudozeros at the distance $\beta(p)$

  $\longrightarrow$ a qualitative analysis of the result

  $\longrightarrow$ a visualization of the result

Pseudozero set for monic polynomials

Perturbation : Neighborhood of polynomial p

$$N_\varepsilon(p) = \{\widehat{p} \in \mathcal{M}_n : \|p - \widehat{p}\| \leqslant \varepsilon\}\,.$$

Definition of the ε-pseudozero set :

$$Z_\varepsilon(p) = \{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\}\,.$$

$\|\cdot\|$ is the 2-norm on the vector of the coefficients of p

The ε-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|\underline{z}\|} \leqslant \varepsilon \right\},$$

where $\underline{z} = (1, z, \ldots, z^{n-1})$

Another characterization of $Z_\varepsilon(p)$

Let us denote $h_{p,\varepsilon} : \mathbf{R}^2 \to \mathbf{R}$, the function

$$h_{p,\varepsilon}(x,y) = |p(x+iy)|^2 - \varepsilon^2 \sum_{j=0}^{n-1} (x^2+y^2)^j.$$

Then one has

$$Z_\varepsilon(p) = \{(x,y) \in \mathbf{R}^2 : h_{p,\varepsilon}(x,y) \leqslant 0\}$$

$\Longrightarrow$ $h_\varepsilon(\cdot, y)$ et $h_\varepsilon(x, \cdot)$ are polynomials of degree $2n$.

Theorem. *The equation $h_{p,\varepsilon}(0,y) = 0$ has a real solution y if and only if $\beta(p) \leqslant \varepsilon$.*

Algorithm (bisection)

Require : a stable polynomial p and a tolerance τ

Ensure : a number α such that $|\alpha - \beta(p)| \leqslant \tau$

$\gamma := 0, \quad \delta := \|p - z^n\|$

while $|\gamma - \delta| > \tau$ **do**

  $\varepsilon := \frac{\gamma+\delta}{2}$

  if the equation $h_{p,\varepsilon}(0, y) = 0$ has a real solution **then**

    $\delta := \varepsilon$

  else

    $\gamma := \varepsilon$

  end if

end while

return $\alpha = \frac{\gamma+\delta}{2}$

Numerical simulation

For $p(z) = z + 1$, the algorithm gives $\beta(p) \approx 0.999996$

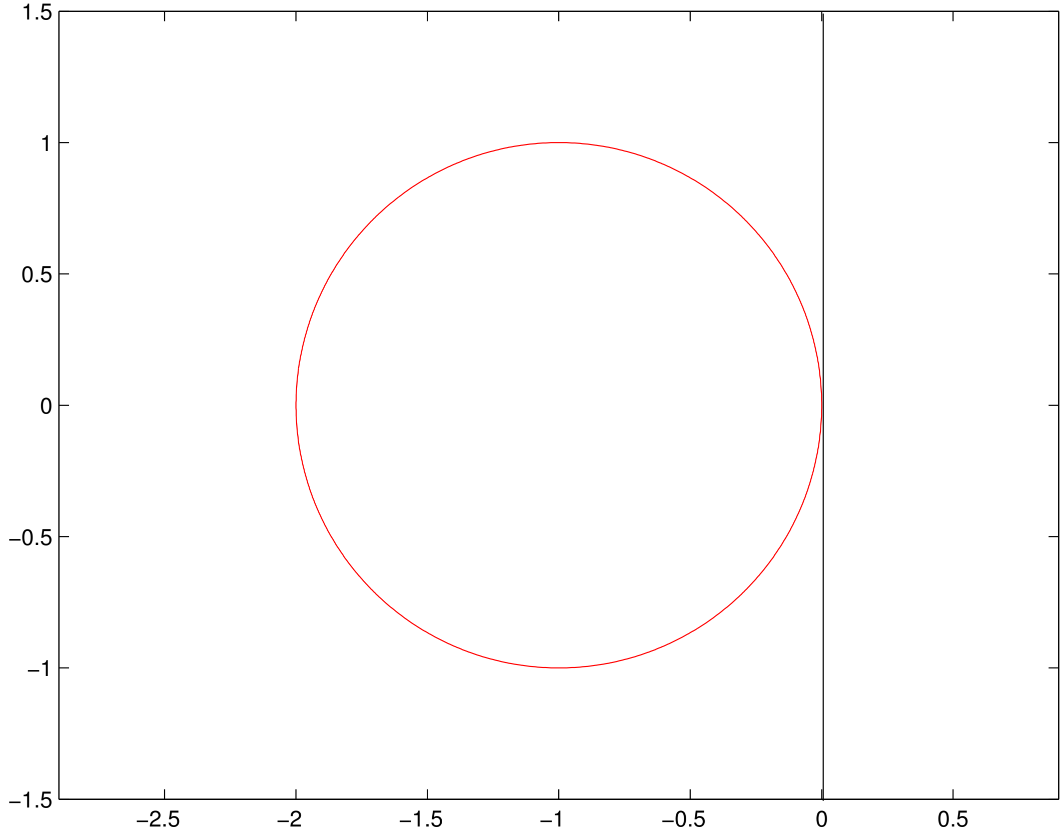


FIG. 1: $\beta(p)$-pseudozero set of $p(z) = z + 1$

Numerical simulation (contd)

For $p(z) = z^2 + z + 1/2$, the algorithm gives $\beta(p) \approx 0.485868$

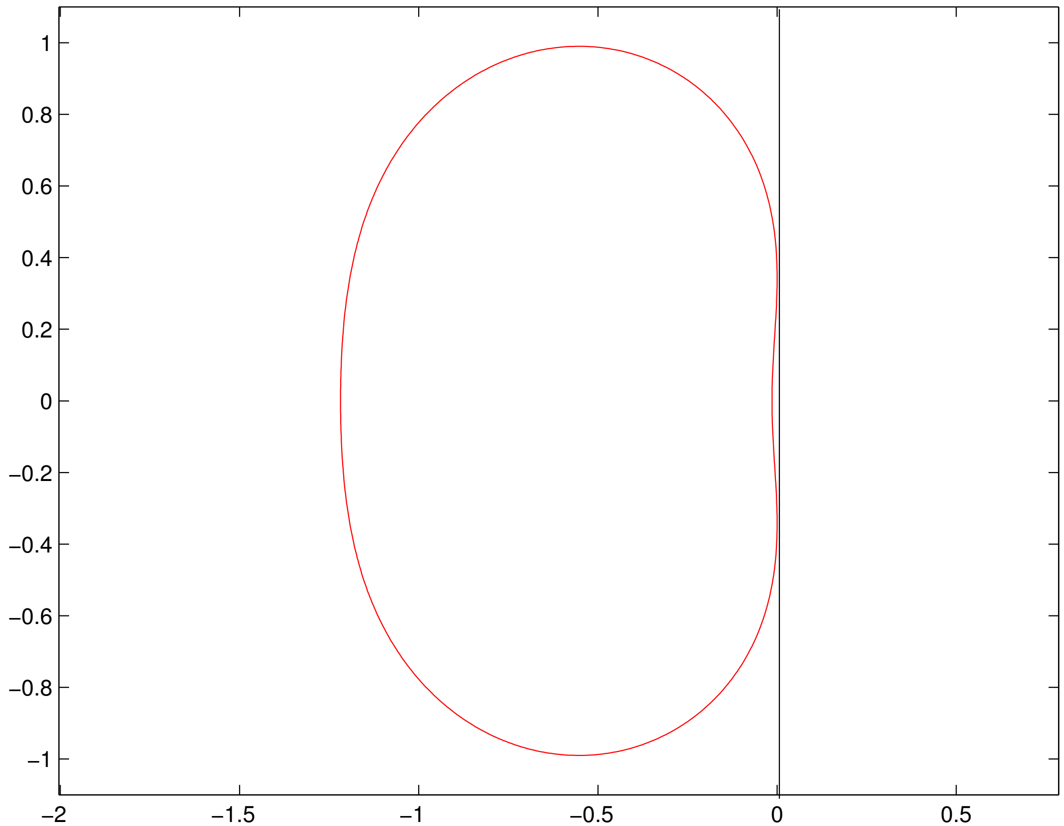


FIG. 2: $\beta(p)$-pseudozero set of $p(z) = z^2 + z + 1/2$

Numerical simulation (contd)

For $p(z) = z^3 + 4z^2 + 6z + 4$, the algorithm gives $\beta(p) \approx 2.610226$

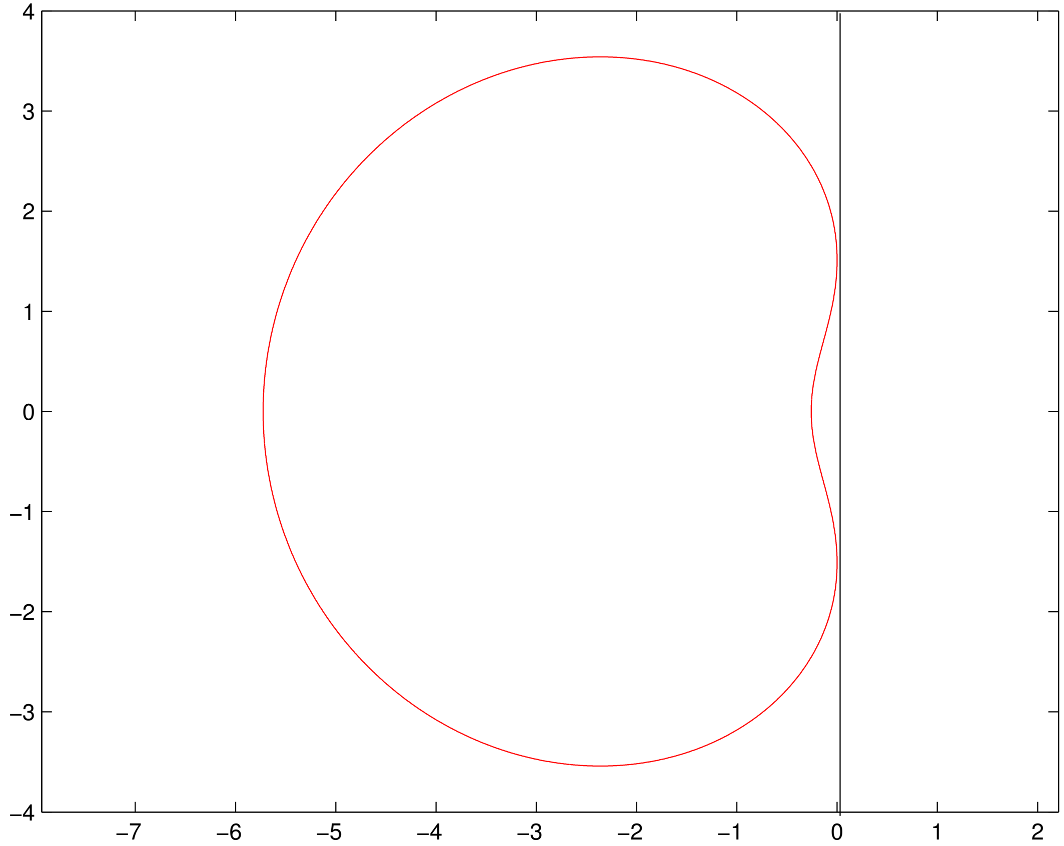


FIG. 3: $\beta(p)$-pseudozero set of $p(z) = z^3 + 4z^2 + 6z + 4$

Conclusion

Pseudozero set provides

- a better understanding of the effect of coefficient perturbations
- some applications for robust stability