Accurate Simple Zeros of Polynomials

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We examine the local behavior of the Newton's method in floating point arithmetic for the computation of a simple zero of a polynomial assuming that a good initial approximation is available. We allow an extended precision (twice the working precision, namely twice the number of digits of the working precision) in the computation of the residual. For that we use the compensated Horner scheme (`CompHorner`) which satisfies the following error bound

$$|\mathtt{CompHorner}(p,x) - p(x)| \le \mathtt{eps}|p(x)| + \gamma_{2n}^2 \tilde{p}(x),$$

where $p(x) = \sum\limits_{i=0}^{n} a_i x^i$, $\tilde{p}(x) = \sum\limits_{i=0}^{n} |a_i||x|^i$, and `eps` is the relative rounding error (for example $\mathtt{eps} = 2^{-53}$ in IEEE 754 double precision).

We use the following accurate Newton's algorithm.

Algorithm 1. *Accurate Newton's method*

$$x_0 = \xi$$
$$x_{i+1} = x_i - \frac{\mathtt{CompHorner}(p, x_i)}{p'(x_i)}$$

We assume that we already know that the root we are looking for belongs to $[a,b]$ with $a, b \in \mathbb{R}$ and we denote $\beta = \max\limits_{x \in [a,b]} |p'(x)|$.

We prove that, for a sufficient number of iterations, the zero is as accurate as if computed with twice the working precision.

Theorem 1. *Assume that there is an x such that $p(x) = 0$ and $p'(x) \neq 0$ is not too small. Assume also that*

$$\mathtt{eps} \cdot \mathrm{cond}(p,x) \le 1/8 \text{ for all } i,$$

where $\mathrm{cond}(p,x) = \tilde{p}(|x|)/|p(x)|$. Then, for all x_0 such that

$$\beta \cdot |p'(x)^{-1}| \cdot |x_0 - x| \le 1/8,$$

Newton's method in floating point arithmetic generates a sequence of $\{x_i\}$ whose relative error decreases until the first i for which

$$\frac{|x_{i+1} - x|}{|x|} \approx \mathtt{eps} + \gamma_{2n}^2 \,\mathrm{cond}(p,x).$$

We provide numerical experiments confirming this. Concerning the number of iterations needed for the accurate Newton's algorithm, it is not clear whether it is the same as the classic Newton's algorithm.

References:

[**1**] Graillat, S.; *Accurate simple zeros of polynomials in floating point arithmetic*, *Comput. Math. Appl.*, 2008. 10 pages, to appear.

[**2**] Graillat, S., Langlois, Ph., Louvet, N.; *Compensated Horner scheme*, Research Report 04, Équipe de recherche DALI, Laboratoire LP2A, Université de Perpignan Via Domitia, France (July 2005).

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