Pseudozeros and Pseudospectra

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Outline

1. Pseudozeros
2. Application of pseudozeros
3. Pseudozeros of interval polynomials
4. Pseudozeros of multivariate polynomials
5. Pseudosectra and structures
6. Open problems

Motivations

- Polynomials appear in almost all areas in scientific computing and engineering
- The relationships between industrial applications and polynomial systems solving studied by the European Community Project FRISCO
- Applications in Computer Aided Design and Modeling, Mechanical Systems Design, Signal Processing and Filter Design, Civil Engineering, Robotics, Simulation
- The wide range of use of polynomial systems needs to have fast and reliable methods to solve them
  - symbolic approach based either on the theory of Gröbner basis or on the theory of resultants
  - numeric approach based on iterative methods like Newton's method or homotopy continuation methods
  - recently, hybrid methods, combining both symbolic and numeric methods

Dealing with uncertainties

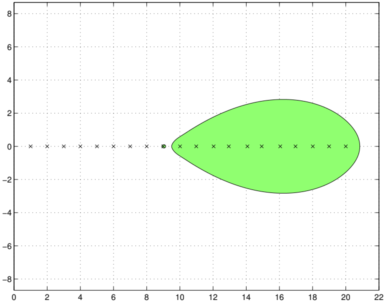
- In practice, from situations arising in science or engineering, the data are known only to a limited accuracy
- Analytical sensitivity analysis introduces a condition number that bounds the magnitudes of the (first order) changes of the roots with respect to the coefficient perturbations
- Continuous sensitivity analysis, introduced by Ostrowski, considers the uncertainty of the coefficients as a continuity problem. The most powerful tool of this last type of methods seems to be the pseudozero set of a polynomial

An example for the univariate case

Computing the zeros of the Wilkinson polynomial of degree 20

$$\begin{aligned} W(x) &= (x-1)(x-2)\cdots(x-20) \\ &= x^{20} - 210x^{19} + \cdots + 20! \end{aligned}$$

Uncertainty of 2^{-23} on the coefficient of x^{19}



Pseudozero set: definition

Perturbation :

Neighborhood of polynomial p

$$N_\varepsilon(p) = \left\{\widehat{p} \in \mathbb{C}_n[z] : \|p - \widehat{p}\| \le \varepsilon\right\}.$$

Definition of the ε-pseudozero set:

$$Z_\varepsilon(p) = \left\{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\right\}.$$

$\|\cdot\|$ a norm on the vector of the coefficients of p

This set is formed by the zeros of polynomials "near p".

Pseudozeros: brief survey of existing references

- Mosier (1986): Definition and study form the ∞-norm.
- Hinrichsen and Kelb: *spectral value sets*
- Trefethen and Toh (1994): Study for the 2-norm.
  pseudozeros $\approx$ pseudospectra of the companion matrix.
- Chatelin and Frayssé (1996): propose a Synthesis in *Lectures on Finite Precision Computations* (SIAM)
- Stetter (1999,2004): *Numerical polynomial algebra.* Generalization of the previous works.
- Zhang (2001): Study of the influence of the basis for the 2-norm (condition number of the evaluation).
- Karow (2003): thesis on *Spectral value sets*

Pseudozeros are easily computable

Theorem 1

The ε-pseudozeros set satisfies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C} : |g(z)| := \frac{|p(z)|}{\|\underline{z}\|_*} \leq \varepsilon \right\},$$

where $\underline{z} = (1, z, \dots, z^n)$ *and* $\|\cdot\|_*$ *is the dual norm of* $\|\cdot\|$,

$$\|y\|_* = \sup_{x \neq 0} \frac{|y^* x|}{\|x\|}$$

The nearest polynomial with a given root p_u

Let p be in $\mathbb{C}_n[z]$ and $u \in \mathbb{C}$.

Statement of the problem:

> *Find a polynomial $p_u \in \mathbb{C}_n[z]$ satisfying $p_u(u) = 0$ and such that if there exists a polynomial $q \in \mathbb{C}_n[z]$ with $q(u) = 0$ then we get $\|p - p_u\| \le \|p - q\|$.*

We are looking for:

- an expression of p_u;
- uniqueness of p_u.

Computation of p_u

Let us denote $\underline{u} := (1, u, u^2, \ldots, u^n) \in \mathbb{C}^{n+1}$.
There exists $d \in \mathbb{C}^{n+1}$ satisfying ${}^t d\underline{u} = \|\underline{u}\|_*$ et $\|d\| = 1$.
Let us define the polynomials r and p_u by

$$\begin{aligned} r(z) &= \sum_{k=0}^{n} r_k z^k \quad \text{with} \quad r_k = d_k, \\ p_u(z) &= p(z) - \frac{p(u)}{r(u)} r(z). \end{aligned}$$

p_u is the nearest polynomial of p with root u.

Uniqueness of p_u

A sufficient condition for uniqueness :

Theorem 2

If the norm $\|\cdot\|$ is strictly convex then p_u is unique.

It is the case, for example, for the norms $\|\cdot\|_p$ for $1 < p < \infty$.

We do not have unicity for $\|\cdot\|_1$ and $\|\cdot\|_\infty$. For $p(z) = 1 + z$

	$\|\cdot\|_1, \quad u = 1$		$\|\cdot\|_\infty, \quad u = 0$	
p_u	$p_1^{(1)}(z) = 0$	$p_1^{(2)}(z) = \frac{1}{3}(1-z)$	$p_0^{(1)}(z) = z$	$p_0^{(2)}(z) = \frac{1}{2}z$
$p - p_i$	$z - 1$	$\frac{4}{3}z - \frac{2}{3}$	1	$\frac{1}{2}z + 1$
$\|p - p_i\|$	2	2	1	1

Algorithm of computation

Algorithm to draw the ε-pseudozero set:

1. We mesh a square containing all the roots of p (MATLAB command: `meshgrid`).
2. We compute $g(z) := \frac{|p(z)|}{\|\underline{z}\|_*}$ for all the nodes z in the grid.
3. We draw the contour level $|g(z)| = \varepsilon$ (MATLAB command: `contour`).

Problems :

- Find a square containing all the roots of p and all the pseudozeros.
- Find a grid step that separates all the roots.

Choice of the grid

Let p be a unitary polynomial of degree n and $\{z_i\}$ the set of its n roots. Let us denote $r = \max_{i=1;\ldots;n} |z_i|$. We have

$$r \leq \max\{1, \sum_{k=1}^{n} |p_k|\}.$$

Let us denote $R := \max\{1, \sum_{i=1}^{n} |p_i| + n\varepsilon\}$. We can prove (in $\|\cdot\|_p$)

$Z_\varepsilon(p) \subset B(0, R)$ the closed ball of centre 0 and radix R.

Complexity of drawing pseudozero set

Let L be the length of the square and h the step of discretization. The evaluation of $g(z) = \frac{|p(z)|}{\|\underline{z}\|_*}$ needs

- the evaluation of polynomial p, that can be done in $\mathcal{O}(n)$,
- the computation of the norm of a vector (the complexity depends on the norm).

Let us denote $\mathcal{O}(\|\cdot\|_*)$ this complexity. The complexity of the algorithm to draw the pseudozero set is

$$\boxed{\mathcal{O}((L/h)^2(n+\|\cdot\|_*))}.$$

L and h depend on n but also on the polynomial coefficients.

A famous example

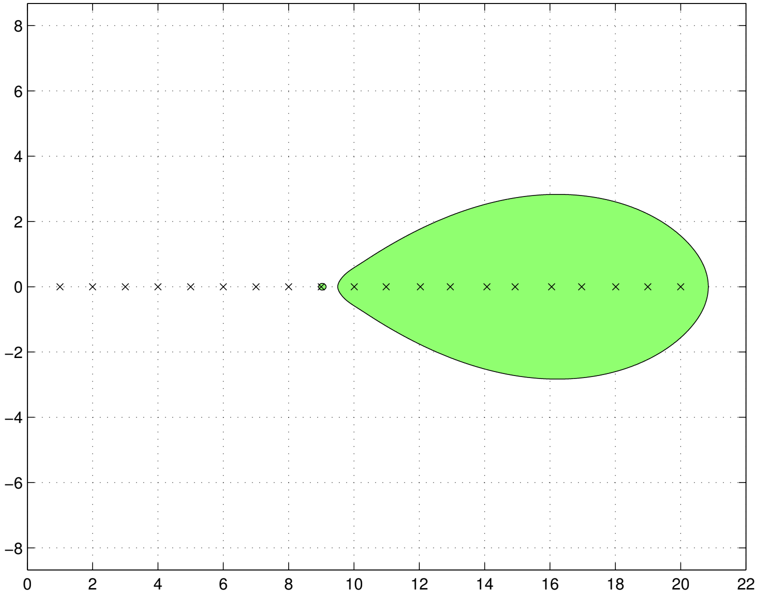
Pseudozero set of the *Wilkinson* polynomial

$$\begin{aligned} W_{20} &= (z-1)(z-2)\cdots(z-20), \\ &= z^{20} - 210z^{19} + \cdots + 20!. \end{aligned}$$

We perturb only the coefficient of z^{19} with $\varepsilon = 2^{-23}$.
One use the weighted-norm $\|\cdot\|_\infty$:

$$\|p\|_\infty = \max_i \frac{|p_i|}{m_i} \text{ with } m_i \text{ non negative}$$

with $m_{19} = 1$, $m_i = 0$ otherwise and the convention $m/0 = \infty$ if $m > 0$ and $0/0 = 0$.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22

Interests of pseudozeros

Pseudozero set provides:

- a qualitative study of polynomials
- a better understanding of the results of polynomial algorithms
- a use of polynomials with coefficients known to a certain accuracy.

Drawback

- the cost

Pseudozeros of real polynomials

If $p \in \mathbb{R}_n[x]$, we define

$$N_\varepsilon(p) := \{q \in \mathbb{R}_n[x] : \|p - q\| \leq \varepsilon\}.$$

Two cases :

- we seek the real pseudozeros: the same as the complex case;
- we seek all the complex non real pseudozeros.

We define the pseudozero set by

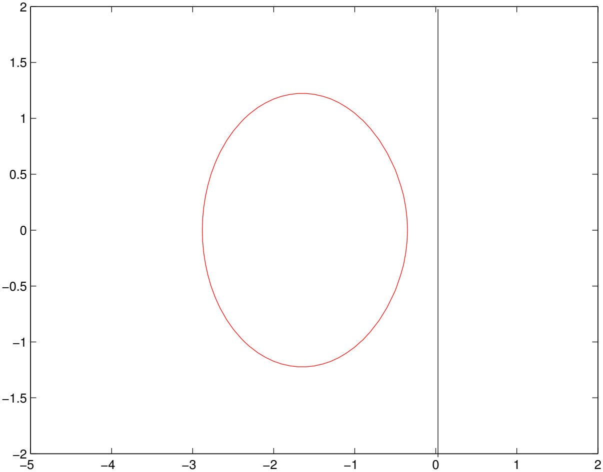
$$Z_\varepsilon(p) := \{z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\}.$$

$Z_\varepsilon(p)$ is symmetrical with respect to the real axis.

Other applications of pseudozeros

Hurwitz robust stability in control theory

Hurwitz stability: Real part of roots of $p < 0$.
ε-pseudozero set of $p(z) = (z+1)^2$ for $\varepsilon = 0.4$.



Computation of stability radius

$\mathscr{P}_n$: polynomials of $\mathbb{C}[X]$ of degree less or equal than n
$\mathscr{M}_n$: monic polynomials of $\mathscr{P}_n$
$\|\cdot\|$: the 2-norm of the coefficients of a polynomial

Definition 1

A polynomial is said to be stable if all the roots have negative real part and unstable otherwise (Hurwitz stability).

The function *abscissa* $a : \mathscr{P} \to \mathbb{R}$ is defined by

$$a(p) = \max\{\mathrm{Re}(z) : p(z) = 0\}.$$

A polynomial p is stable $\iff a(p) < 0$

Motivation

En control theory, a transfer function can be written as $H(p) = \frac{N(p)}{D(p)}$ where N and D are polynomials.

The system is stable if D is a stable polynomial .

Question : if D is stable, is it far from unstable system?

Problem : Find the distance to the nearest unstable system.
(we assume that D is monic)

Statement of the problem

Stability radius $\beta(p)$: distance of the polynomial $p \in \mathcal{M}_n$ from the set of monic unstable polynomials.

$$\beta(p) = \min\{\|p - q\| : q \in \mathcal{M}_n \text{ and } a(q) \geq 0\}.$$

Statement of the problem:

Given a polynomial $p \in \mathcal{M}_n$, compute $\beta(p)$.

Solution

Tools

- an explicit formula giving the pseudozeros
- the continuous dependency of the roots with respect to the polynomial coefficients
- the Sturm sequences to count the real roots

The results

- a algorithm calculating $\beta(p)$ with an arbitrary tolerance τ
- a drawing showing the pseudozeros at the distance $\beta(p)$
  $\longrightarrow$ enable a qualitative analysis of the result
  $\longrightarrow$ visualization of the result

Another characterization of $Z_\varepsilon(p)$

Let us denote $h_{p,\varepsilon} : \mathbb{R}^2 \to \mathbb{R}$ the function defined by

$$h_{p,\varepsilon}(x,y) = |p(x+iy)|^2 - \varepsilon^2 \sum_{j=0}^{n-1} (x^2+y^2)^j.$$

Then one has

$$\boxed{Z_\varepsilon(p) = \{(x,y) \in \mathbb{R}^2 : h_{p,\varepsilon}(x,y) \leq 0\}}$$

$\Longrightarrow$ $h_\varepsilon(\cdot, y)$ et $h_\varepsilon(x,\cdot)$ are polynomials of degree $2n$.

Theoretical results

Proposition 1

The function abscissa

$$a : \mathscr{P}_n \to \mathbb{R}$$

defined by $a(p) = \max\{\mathrm{Re}(z) : p(z) = 0\}$ *is continuous on* $\mathscr{M}_n$.

Proposition 2

One has the following relation

$$\beta(p) = \min\{\|p - q\| : q \in \mathscr{M}_n \text{ and } a(q) = 0\}.$$

Theorem 3

The equation $h_{p,\varepsilon}(0, y) = 0$ *has a real solution* y *if and only if* $\beta(p) \le \varepsilon$.

Algorithm (bisection)

Require: a stable polynomial p and a tolerance τ
Ensure: a number α such that $|\alpha - \beta(p)| \le \tau$

$\gamma := 0, \quad \delta := \|p - z^n\|$
while $|\gamma - \delta| > \tau$ **do**
    $\varepsilon := \frac{\gamma+\delta}{2}$
    if the equation $h_{p,\varepsilon}(0, y) = 0,\ y \in \mathbb{R}$ has a solution **then**
        $\delta := \varepsilon$
    else
        $\gamma := \varepsilon$
    end if
end while
return $\alpha = \frac{\gamma+\delta}{2}$

Numerical simulation

For the polynomial $p(z) = z + 1$, the algorithm gives $\beta(p) \approx 0.999996$

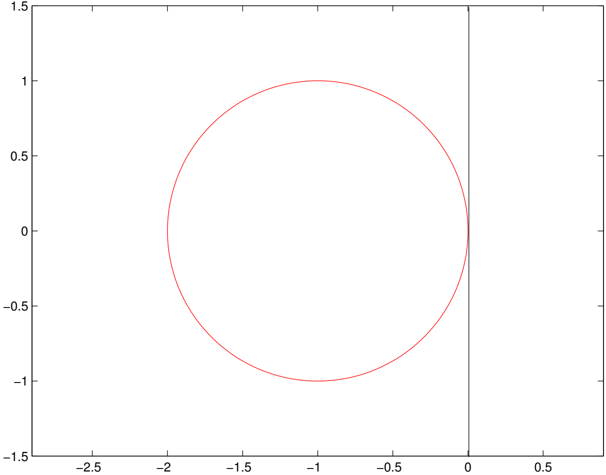


Figure : $\beta(p)$-pseudozero set of $p(z) = z + 1$

Numerical simulation (contd)

For the polynomial $p(z) = z^3 + 4z^2 + 6z + 4$, the algorithm gives $\beta(p) \approx 2.61022$

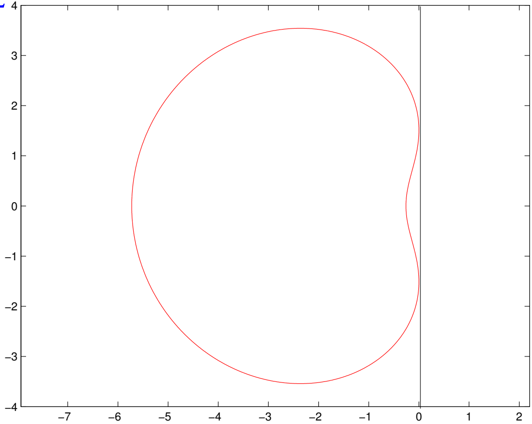


Figure : $\beta(p)$-pseudozero set of $p(z) = z^3 + 4z^2 + 6z + 4$

Pseudozero set of interval polynomials

Interval polynomial

An interval polynomial: polynomial whose coefficients are real intervals.
We denote by $\mathbb{IR}[z]$ the set of interval polynomials and by $\mathbb{IR}_n[z]$ the set of interval polynomials with degree at most n.
Let $p \in \mathbb{IR}_n[z]$. We can write

$$p(z) = \sum_{i=0}^{n} [a_i, b_i] z^i.$$

The zeros of an interval polynomial is the set

$$\mathbb{Z}(p) := \{z \in \mathbb{C} : \text{there exist } m_i \in [a_i, b_i],\, i = 0 : n \text{ such that} \sum_{i=0}^{n} m_i z^i = 0\}.$$

$\Longrightarrow$ Compute $\mathbb{Z}(p)$.

Definition of real pseudozero set

Let $p = \sum_{i=0}^{n} p_i z^i$ be a polynomial of $\mathbb{R}_n[z]$

Perturbations :

Real neighborhood of p

$$N_\varepsilon^R(p) = \left\{ \widehat{p} \in \mathbb{R}_n[z] : \| p - \widehat{p} \| \le \varepsilon \right\}.$$

Definition of the real ε-pseudozero set

$$Z_\varepsilon^R(p) = \left\{ z \in \mathbb{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon^R(p) \right\}.$$

Computation of the real pseudozero set

Theorem:
The real ε-pseudozero set satisfies

$$Z_\varepsilon^R(p) = Z(p) \cup \left\{ z \in \mathbb{C} \setminus Z(p) : h(z) := d(G_R(z), \mathbb{R} G_I(z)) \geq \frac{1}{\varepsilon} \right\},$$

where d is defined for $x, y \in \mathbb{R}^{n+1}$ by

$$d(x, \mathbb{R}y) = \inf_{\alpha \in \mathbb{R}} \|x - \alpha y\|_*$$

and where $G_R(z)$, $G_I(z)$ are the real and imaginary part of

$$G(z) = \frac{1}{p(z)} (1, z, \ldots, z^n)^T, \; z \in \mathbb{C} \setminus Z(p)$$

Can be viewed as a special case of *spectral value set* [Karow 03]

What for $\mathbb{R} \cap Z_\varepsilon^R(p)$?

Lemma 1

Given $z \in \mathbb{R}$, z belongs to $Z_\varepsilon^R(p)$ if and only if z belongs to $Z_\varepsilon(p)$.

Draw the complex pseudozero set or the real pseudozero set on the real axis is similar.

Some properties

The function d defined for $x, y \in \mathbb{R}^{n+1}$ by

$$d(x, \mathbb{R}y) = \inf_{\alpha \in \mathbb{R}} \|x - \alpha y\|_*$$

satisfies

$$d(x, \mathbb{R}y) = \begin{cases} \sqrt{\|x\|_2^2 - \frac{\langle x, y\rangle^2}{\|y\|_2^2}} & \text{if } y \neq 0, \\ \|x\|_2 & \text{if } y = 0 \end{cases} \qquad \text{for the norm} \|\cdot\|_2$$

$$d(x, \mathbb{R}y) = \begin{cases} \min_{\substack{i=0:n \\ y_i \neq 0}} \|x - (x_i/y_i)y\|_1 & \text{if } y \neq 0, \\ \|x\|_1 & \text{if } y = 0 \end{cases} \qquad \text{for the norm } \|\cdot\|_\infty$$

Some properties (cont'd)

Proposition 3

The real ε-pseudozero set $Z^R_\varepsilon(p)$ is symmetric with respect to the real axis.

Proposition 4

The real ε-pseudozero set $Z^R_\varepsilon(p)$ is included in the complex ε-pseudozero set.

Algorithm to draw real pseudozero set

Drawing of real ε-pseudozero set:

1. We mesh a square containing all the roots of p (MATLAB command: `meshgrid`).
2. We compute $h(z) := d(G_R(z), \mathbb{R}G_I(z))$ for all the nodes z in the grid.
3. We draw the contour level $|h(z)| = \frac{1}{\varepsilon}$ (MATLAB command: `contour`).

Pseudozero set with weighted norm

$$p(z) = \sum_{i=0}^{n} p_i z^i.$$

- identification of p with the vector $(p_0, p_1, \ldots, p_n)^T$
- $d := (d_0, \ldots, d_n)^T \in \mathbb{R}^{n+1}$ represents the weight of the coefficients of p
- $\|\cdot\|_{\infty,d}$ defined by

$$\|p\|_{\infty,d} = \max_{i=0:n}\{|p_i|/|d_i|\}.$$

- Its dual norm is

$$\|x\|_{1,d} := \sum_{i=0}^{n} |d_i||x_i|.$$

Zeros of interval polynomials and real pseudozero set

Let us denote p_c the central polynomial defined by

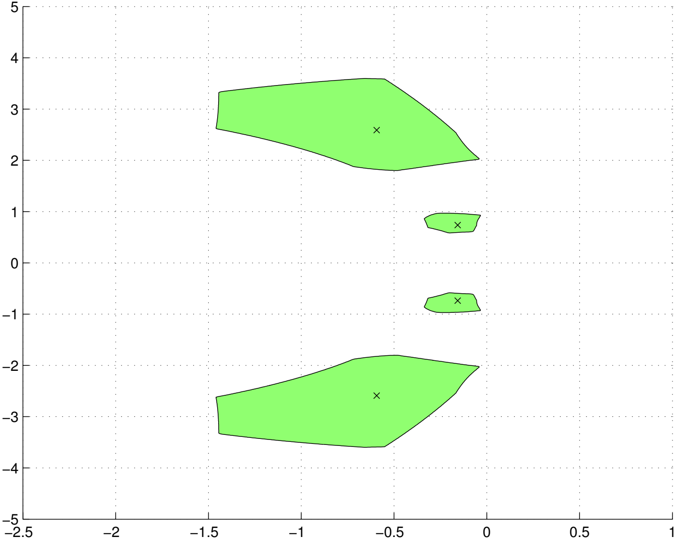
$$p_c(z) = \sum_{i=0}^{n} c_i z^i,$$

with $c_i = (a_i + b_i)/2$.
Let us denote $d_i := (b_i - a_i)/2$.
Proposition:
With the notation above, we have

$$\mathbb{Z}(p) = Z_\varepsilon^R(p_c) \text{ with } \varepsilon = 1.$$

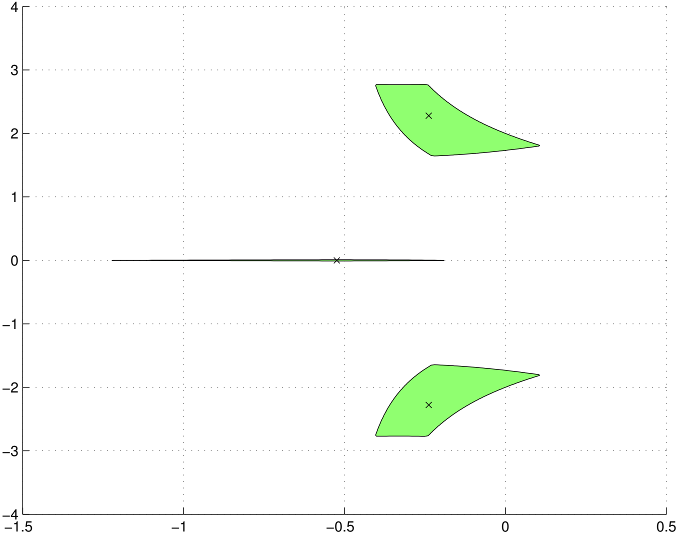
Example 1

$$p(z) = [1,2]z^4 + [3.2,3]z^3 + [10,14]z^2 + [3,5\sqrt{2}]z + [5,7]$$



Example 2

$p(z) = z^3 + z^2 + [3,8]z + [1.5,4]$



Problem: choice of the grid

Lemma :
Let $p(z) = \sum_{i=0}^{n} [a_i, b_i] z^i$ an interval polynomial and

$$R := 1 + \frac{\max_{i=0:n}\{\max\{|a_i|, |b_i|\}\}}{\min\{|a_n|, |b_n|\}}.$$

Then

$$\mathbb{Z}(p) \subset B(O, R),$$

where $B(O, R)$ the ball in $\mathbb{C}$ of centre O and radius R.

Problems: discontinuities

Lemma [Hinrichsen et Kelb]:
The function

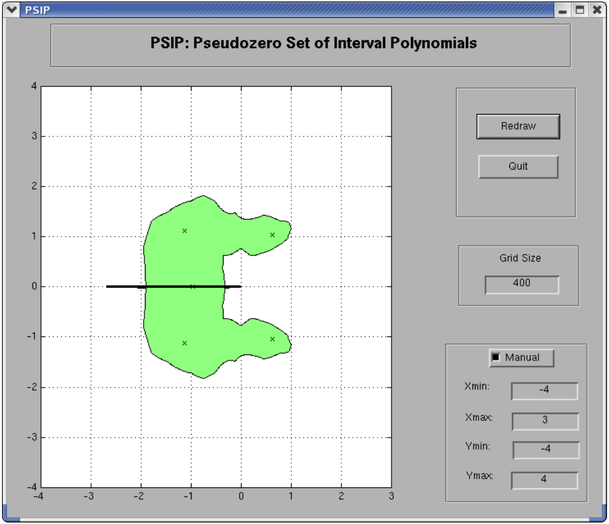
$$d : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}_+, \quad (x, y) \mapsto d(x, \mathbb{R}y)$$

is continue for all (x, y) with $y \neq 0$ or $x = 0$ and discontinue for all $(x, 0) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, $x \neq 0$.
$\Longrightarrow$ Those discontinuties imply some difficulties for drawing near the real axis.

Solution : on the real axis, we draw complex pseudozero set.

Presentation of PSIP

A tool to draw zeros of interval polynomials



Presentation of PSIP (cont'd)

- a graphical interface for MATLAB (version 6.5 (R13))
- computation of grid that contains all the zeros
- possibilities of zoom and mesh refinement

Limitations :

- problem if the leading interval contains 0
- problems with discontinuities

Pseudozero set of multivariate polynomials

Definitions (1/3)

A monomial in the n variables $z_1, \ldots, z_n$ is the power product

$$z^j := z_1^{j_1} \cdots z_n^{j_n}, \quad \text{with } j = (j_1, \ldots, j_n) \in \mathbb{N}^n;$$

j is the exponent and $|j| := \sum_{\sigma=1}^{n} j_\sigma$ the *degree* of the monomial z^j.

Definition 2

A complex (real) polynomial in n variables is a finite linear combination of monomials in n variables with coefficients from $\mathbb{C}$ (from $\mathbb{R}$),

$$p(z) = p(z_1, \ldots, z_n) = \sum_{(j_1,\ldots,j_n)\in J}^{n} a_{j_1 \cdots j_n} z_1^{j_1} \cdots z_n^{j_n} = \sum_{j\in J} a_j z^j.$$

$\mathscr{P}^n(\mathbb{C})$ ($\mathscr{P}^n(\mathbb{R})$) represents the set of all complex (real) polynomials in n variables.

Definitions (2/3)

Given $p = \sum_{j \in J} a_j z^j \in \mathscr{P}^n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

$\longrightarrow$ $|J|$ the number of elements of J

If $|J| = M$ and let $\|\cdot\|$ be a norm on $\mathbb{K}^M$

$\longrightarrow$ $\|p\|$ is the norm of the vector $a = (\dots, a_j, \dots, j \in J)$

Given a norm $\|\cdot\|$ on $\mathbb{K}^N$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the dual norm is defined by $\|x\|_* := \sup_{\|y\|=1} |y^T x|$.

Given a vector $x \in \mathbb{K}^N$, there exists a dual vector $y \in \mathbb{K}^N$ with $\|y\| = 1$ satisfying $x^T y = \|x\|_*$.

Norms	Dual norms
$\|x\|_1 := \sum_j \|x_j\|$	$\|x\|_1^* = \max_j \|x_j\| = \|x\|_\infty$
$\|x\|_2 := (\sum_j \|x_j\|^2)^{1/2}$	$\|x\|_2^* = (\sum_j \|x_j\|^2)^{1/2} = \|x\|_2$
$\|x\|_\infty := \max_j \|x_j\|$	$\|x\|_\infty^* = \sum_j \|x_j\| = \|x\|_1$

Definitions (3/3)

Given $\varepsilon > 0$, the ε-neighborhood $N_\varepsilon(p)$ of the polynomial $p \in \mathscr{P}^n(\mathbb{K})$ is the set of all polynomials of $\mathscr{P}^n(\mathbb{K})$ with $\widetilde{p} = \sum_{j \in \widetilde{J}} \widetilde{a}_j z^j \in \mathscr{P}^n(\mathbb{K})$ with support $\widetilde{J} \subset J$ and $\|\widetilde{p} - p\| \leq \varepsilon$.

Definition 3

A value $z \in \mathbb{K}^n$ is an ε-pseudozero of a polynomial $p \in \mathscr{P}^n$ if it is a zero of some polynomial $\widetilde{p}$ in $N_\varepsilon(p)$.

Definition 4

The ε-pseudozero set of a polynomial $p \in \mathscr{P}^n$ (denoted by $Z_\varepsilon(p)$) is the set of all the ε-pseudozeros,

$$Z_\varepsilon(p) := \{z \in \mathbb{K}^n : \exists \widetilde{p} \in N_\varepsilon(p), \ \widetilde{p}(z) = 0\}.$$

Pseudozeros of complex multivariate polynomials (1/2)

Theorem 4 (Stetter)

The complex ε-pseudozero set of $p = \sum_{j \in J} a_j z^j \in \mathscr{P}^n(\mathbb{C})$ verifies

$$Z_\varepsilon(p) = \left\{ z \in \mathbb{C}^n : g(z) := \frac{|p(z)|}{\|\mathbf{z}\|_*} \leq \varepsilon \right\}$$

where $\mathbf{z} := (\ldots, |z|^j, \ldots, j \in J)^T$.

Pseudozeros of complex multivariate polynomials (2/2)

Corollary 1 (Stetter)

The complex ε-pseudozero set of $P = \{p_1, \dots, p_k\}$, $k \in \mathbb{N}$ verifies

$$Z_\varepsilon(P) = \left\{ z \in \mathbb{C}^n : \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \leq \varepsilon \text{ for } l = 1, \dots, k \right\},$$

where $\mathbf{z_l} := (\dots, |z|^j, \dots, j \in J_l)^T$.

We restrict our attention to situations where P as well as all the systems in $N_\varepsilon(P)$ are 0-dimensional, that is, if the solution of the system is non-empty and finite.

Theorem 5 (Stetter)

Each system $\widetilde{P} \in N_\varepsilon(P)$ has the same number of zeros (counting multiplicities) in a fixed pseudozero set connected component of $Z_\varepsilon(P)$.

Pseudozeros of real multivariate polynomials: definition

A real ε-neighborhood of p is the set of all polynomials of $\mathscr{P}^n(\mathbb{R})$, close enough to p, that is to say,

$$N_\varepsilon^R(p) = \left\{\widetilde{p} \in \mathscr{P}^n(\mathbb{R}) : \|p - \widetilde{p}\| \le \varepsilon\right\}.$$

The real ε-pseudozero set of p is defined to include all the zeros of the real ε-neighborhood of p:

$$Z_\varepsilon^R(p) = \left\{z \in \mathbb{C}^n : \widetilde{p}(z) = 0 \text{ for } \widetilde{p} \in N_\varepsilon^R(p)\right\}.$$

For $\varepsilon = 0$, the pseudozero set $Z_0^R(p)$ is the set of the roots of p we denote $Z(p)$.

Pseudozeros of real multivariate polynomials: computation

Distance of a point $x \in \mathbb{R}^N$ from the linear subspace $\mathbb{R}y = \{\alpha y, \alpha \in \mathbb{R}\}$

$$d(x, \mathbb{R}y) = \inf_{\alpha \in \mathbb{R}} \|x - \alpha y\|_*,$$

Theorem 6

The real ε-pseudozero set of $p = \sum_{j \in J} a_j z^j \in \mathscr{P}^n(\mathbb{R})$ verifies

$$Z_\varepsilon^R(p) = Z(p) \cup \left\{ z \in \mathbb{C}^n \backslash Z(p) : h(z) := d(G_R(z), \mathbb{R}G_I(z)) \geq \frac{1}{\varepsilon} \right\},$$

where $G_R(z)$ and $G_I(z)$ are the real and imaginary parts of

$$G(z) = \frac{1}{p(z)} (\ldots, z^j, \ldots, j \in J)^T, \; z \in \mathbb{C}^n \backslash Z(p).$$

Computing the distance

- computing real ε-pseudozero set $Z_\varepsilon^R(p)$ needs to evaluate the distance $d(G_R(z), \mathbb{R}G_I(z))$.
- the 2-norm $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$ the corresponding inner product

$$d(x, \mathbb{R}y) = \begin{cases} \sqrt{\|x\|_2^2 - \frac{\langle x,y\rangle^2}{\|y\|_2^2}} & \text{if } y \neq 0, \\ \|x\|_2 & \text{if } y = 0. \end{cases}$$

- the ∞-norm,

$$d(x, \mathbb{R}y) = \begin{cases} \min_{\substack{i=0:n \\ y_i \neq 0}} \|x - (x_i/y_i)y\|_1 & \text{if } y \neq 0, \\ \|x\|_1 & \text{if } y = 0. \end{cases}$$

- other p-norm with $p \neq 2, \infty$, no easy computable formula to calculate $d(x, \mathbb{R}y)$.

Real pseudozeros of polynomial systems

Corollary 2

The real ε-pseudozero set of $P = \{p_1, \ldots, p_k\}$, $k \in \mathbb{N}$ *verifies*

$$Z_\varepsilon^R(P) = \bigcap_{l=1}^{k} \left(Z(p_l) \cup \left\{ z \in \mathbb{C}^n \backslash Z(p_l) : d(G_R^l(z), \mathbb{R}G_I^l(z)) \geq \frac{1}{\varepsilon} \right\} \right)$$

where $G_R^l(z)$ *and* $G_I^l(z)$ *are the real and imaginary parts of*

$$G^l(z) = \frac{1}{p_l(z)} (\ldots, z^j, \ldots, j \in J_l)^T,\ z \in \mathbb{C}^n \backslash Z(p_l).$$

Visualization of pseudozero sets (1/5)

- The descriptions of $Z_\varepsilon(P)$ and $Z_\varepsilon^R(P)$ given previously make it possible to compute, plot and visualize pseudozero set of multivariate polynomials.
- The pseudozero set is a subset of $\mathbb{C}^n$ which can only be seen by its projections on low dimensional spaces that is often $\mathbb{C}$.

We have written a MATLAB program to compute and visualize these projections. This program requires the Symbolic Math Toolbox.

Visualization of pseudozero sets (2/5)

For a given $v \in \mathbb{C}^n$, let $Z_\varepsilon(P, j, v)$ be the projection of $Z_\varepsilon(P)$ onto the z_j-space around v. Then, it follows that for $P = \{p_1, \ldots, p_k\}$,

$$Z_\varepsilon(P, j, v) = \left\{ z \in \mathbb{C}^n : z_i = v_i,\ i \neq j,\ \max_{l=1,\ldots,k} \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \leq \varepsilon \right\},$$

where $\mathbf{z_l} := (\ldots, |z|^j, \ldots, j \in J_l)^T$.

One way for visualizing $Z_\varepsilon(P, j, v)$ is to plot the values of the projection of

$$\mathrm{ps}(z) := \log_{10}\left(\max_{l=1,\ldots,k} \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \right)$$

over a set of grid points around v in z_j-space.

Visualization of pseudozero sets (3/5)

In the same way, we define for a given $v \in \mathbb{C}^n$, $Z_\varepsilon^R(P,j,v)$ by the projection of $Z_\varepsilon^R(P)$ onto the z_j-space around v. It follows that for $P = \{p_1, \ldots, p_k\}$,

$$Z_\varepsilon^R(P,j,v) = \left\{ z \in \mathbb{C}^n : z_i = v_i,\ i \neq j,\ \max_{l=1,\ldots,k} d(G_R^l(z), \mathbb{R}G_I^l(z))^{-1} \leq \varepsilon \right\}$$

where $G_R^l(z)$ and $G_I^l(z)$ are the real and imaginary parts of

$$G^l(z) = \frac{1}{p_l(z)}(\ldots, z^j, \ldots, j \in J_l)^T,\ z \in \mathbb{C}^n \backslash Z(p_l).$$

One way for visualizing $Z_\varepsilon^R(P,j,v)$ is still to plot the values of the projection of

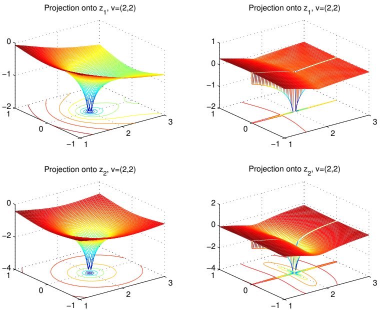
$$\mathrm{ps}^R(z) := \log_{10}\left(\max_{l=1,\ldots,k} d(G_R^l(z), \mathbb{R}G_I^l(z))^{-1} \right)$$

over a set of grid points around v in z_j-space.

Visualization of pseudozero sets (4/5)

We examine the following system using the 2-norm: two unit balls intersection at $(2,2)$,

$$P_1 = \begin{cases} p_1 = (z_1-1)^2 + (z_2-2)^2 - 1, \\ p_2 = (z_1-3)^2 + (z_2-2)^2 - 1. \end{cases}$$



Projections of the complex pseudozero set (on the left) and the real pseudozero set (on the right) of P_1

Visualization of pseudozero sets (5/5)

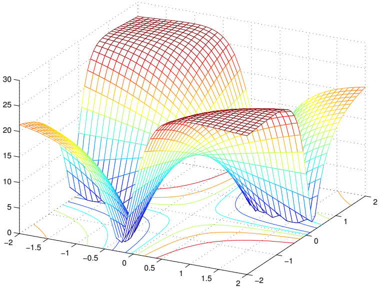
We can be only interested in the real zeros of a polynomial systems. In this case, we can only draw $\mathbb{R}^n \cap Z_\varepsilon^R(P)$.

$$P_2 = \begin{cases} p_1 = z_1^2 + z_2^2 - 1, \\ p_2 = 25 z_1 z_2 - 12. \end{cases}$$

We have computed the function

$$g(x,y) = \max_{l=1,2} \frac{p_l(x,y)}{\|\mathbf{z_l}\|_*},$$

with $\mathbf{z_l} := (\ldots, |x+iy|^j, \ldots, j \in J_l)^T$.



Projection of the real pseudozero set of P_2

Pseudospectra of matrices

Why structured matrices?

- Structured matrices are used in various fields such as signal processing, etc.
- Using the structure of a matrix, we get some better properties
- Substantial interest in algorithms for structured problems in recent years
- Growing interest in structured perturbation analysis
- In general perturbation and error analysis for structured solvers are performed with *general* perturbations: for a structured solver nothing else but structured perturbations are *possible*

Our structures

Toeplitz matrices $(t_{i-j})_{i,j=0}^{n-1}$	$\begin{pmatrix} t_0 & t_{-1} & \cdots & t_{1-n} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & t_1 & t_0 \end{pmatrix}$
Hankel matrices $(h_{i,j})_{i,j=0}^{n-1}$	$\begin{pmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \iddots & h_n \\ \vdots & \iddots & \iddots & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n-2} \end{pmatrix}$
Circulant matrices $(v_i)_{i=0}^{n-1}$	$\begin{pmatrix} v_0 & v_{n-1} & \cdots & v_1 \\ v_1 & v_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_{n-1} \\ v_{n-1} & \cdots & v_1 & v_0 \end{pmatrix}$

Number of independant parameters

- In the following table, k represents the number of independant parameters for the different structures

Structure	general	Toeplitz	circulant	Hankel
k	n^2	$2n-1$	n	$2n-1$

Notations

In this talk, we will use the following notation:

struct	Toeplitz, circulant or Hankel
$M_n(\mathbb{C})$	set of complex $n \times n$ matrices
$M_n^{\text{struct}}(\mathbb{C})$	set of structured complex $n \times n$ matrices
$\\|\cdot\\|$	spectral norm
I, I_n	identity matrix (with n rows and columns)
$\sigma_{\min}(A)$	smallest singular value of A
$\Lambda(A)$	spectrum of A

Definition of pseudospectra

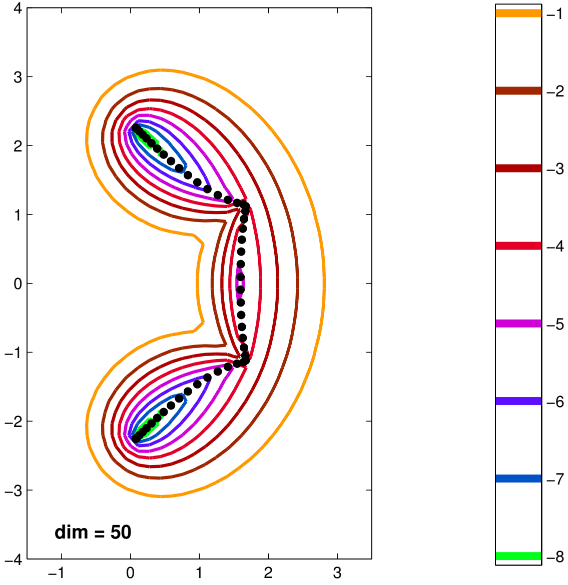
The ε-pseudospectrum of a matrix A, denoted $\Lambda_\varepsilon(A)$, is the subset of complex numbers consisting of all eigenvalues of all complex matrices within a distance ε of A

Definition 5

For a real $\varepsilon > 0$, the ε-pseudospectrum of a matrix $A \in M_n(\mathbb{C})$ is the set

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : z \in \Lambda(X) \text{ where } X \in M_n(\mathbb{C}) \text{ and } \|X - A\| \leq \varepsilon\}.$$

Example of pseudospectra



Distance to singularity

Definition 6

Given a nonsingular matrix $A \in M_n(\mathbb{C})$, we define the distance to singularity by

$$d(A) = \min\{\|\Delta A\| : A + \Delta A \text{ singular}, \Delta A \in M_n(\mathbb{C})\}.$$

Lemma 2 (Gastinel)

Let nonsingular $A \in M_n(\mathbb{C})$. Then we have

$$d(A) = \|A^{-1}\|^{-1}.$$

Characterisation of pseudospectra

Theorem 7 (Trefethen)

The following assertions are equivalent

(i) $\Lambda_\varepsilon(A)$ *is the* ε*-pseudospectrum of a matrix* A

(ii) $\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : \|(zI - A)^{-1}\| \geq \varepsilon^{-1}\}$

(iii) $\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : \sigma_{min}(zI - A)\| \leq \varepsilon\}$

(iv) $\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : d(zI - A) \leq \varepsilon\}$

Definition of structured pseudospectra

The structured ε-pseudospectrum of a matrix A, denoted $\Lambda_\varepsilon^{\text{struct}}(A)$, is the subset of complex numbers consisting of all eigenvalues of all complex structured matrices within a distance ε of A

Definition 7

For a real $\varepsilon > 0$, the structured ε-pseudospectrum of a matrix $A \in M_n^{\text{struct}}(\mathbb{C})$ is the set

$$\Lambda_\varepsilon^{\text{struct}}(A) = \{z \in \mathbb{C} : z \in \Lambda(X) \text{ where } X \in M_n^{\text{struct}}(\mathbb{C}) \\ \text{and } \|X - A\| \leq \varepsilon\}.$$

Structured distance to singularity

Definition 8

Given a nonsingular matrix $A \in M_n^{\text{struct}}(\mathbb{C})$, *we define the structured distance to singularity by*

$$d^{\text{struct}}(A) = \min\{\|\Delta A\| : A + \Delta A \text{ singular}, \Delta A \in M_n^{\text{struct}}(\mathbb{C})\}.$$

Theorem 8 (Rump)

Let nonsingular $A \in M_n^{\text{struct}}(\mathbb{C})$ *with struct being Toeplitz , Hankel or circulant. Then we have*

$$d^{\text{struct}}(A) = d(A) = \|A^{-1}\|^{-1}.$$

Characterisation of structured pseudospectra

Lemma 3

Given $\varepsilon > 0$ and $A \in M_n^{\text{struct}}(\mathbb{C})$ with struct Toeplitz or circulant, the structured ε-pseudospectrum satisfies

$$\Lambda_\varepsilon^{\text{struct}}(A) = \{z \in \mathbb{C} : d^{\text{struct}}(A - zI) \leq \varepsilon\}.$$

Theorem 9

Given $\varepsilon > 0$ and $A \in M_n^{\text{struct}}(\mathbb{C})$ with struct Toeplitz or circulant, the ε-pseudospectrum and the structured ε-pseudospectrum satisfy

$$\Lambda_\varepsilon^{\text{struct}}(A) = \Lambda_\varepsilon(A).$$

What for others linear structures?

We do not have equality for Hermitian and skew-Hermitian structures.
For example for Hermitian structure we always have $\Lambda_\varepsilon^{\text{herm}}(A) \subsetneq \mathbb{R}$ whereas one can find an Hermitian matrix such that $\Lambda_\varepsilon(A) \nsubseteq \mathbb{R}$.

The polynomial eigenvalue problem

Problem 10

Find the solutions $(x, \lambda) \in \mathbb{C}^n \times \mathbb{C}$ *of*

$$P(\lambda)x = 0,$$

where

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n(\mathbb{C})$, $k = 0 : m$

If $x \neq 0$ then λ is called an eigenvalue and x the corresponding eigenvector. The set of eigenvalues of P is denoted $\Lambda(P)$. We assume that P has only finite eigenvalues (and pseudoeigenvalues)

Definition of pseudospectra

Let us define

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n(\mathbb{C})$.

Definition 9

For a given $\varepsilon > 0$, the ε-pseudospectrum of P is the set

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbb{C} : (P(\lambda) + \Delta P(\lambda))x = 0 \textit{ for some } x \neq 0 \\ \textit{with } \|\Delta A_k\| \leq \alpha_k \varepsilon, k = 0 : m\}.$$

The nonnegative parameters $\alpha_1, \ldots, \alpha_m$ allow freedom in how perturbations are measured

Characterisation of pseudospectra

Lemma 4 (Tisseur and Higham (2001))

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbb{C} : d(P(\lambda)) \leq \varepsilon p(|\lambda|)\},$$

where $p(x) = \sum_{k=0}^{m} \alpha_k x^k$.

Definition of structured pseudospectra

We suppose that ΔA_k have a structure belonging to struct. We also suppose that all the matrices A_k and ΔA_k, $k = 0 : n$, belong to $M_n^{\text{struct}}(\mathbb{C})$ for a given structure struct. Let

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n^{\text{struct}}(\mathbb{C})$, $k = 0 : m$ and

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n^{\text{struct}}(\mathbb{C})$. $P(\lambda)$ and $\Delta P(\lambda)$ belong to $M_n^{\text{struct}}(\mathbb{C})$.

Definition 10

We define the structured ε-pseudospectrum of P by

$$\Lambda_\varepsilon^{\text{struct}}(P) = \{\lambda \in \mathbb{C} : (P(\lambda) + \Delta P(\lambda))x = 0 \text{ for some } x \neq 0 \\ \text{with } \Delta A_k \in M_n^{\text{struct}}(\mathbb{C}), \|\Delta A_k\| \leq \alpha_k \varepsilon, k = 0 : n\}.$$

Characterisation of structured pseudospectra

Lemma 5

For struct $\in$ {Toep, circ, Hankel}, *we have*

$$\Lambda_\varepsilon^{\text{struct}}(P) = \{\lambda \in \mathbb{C} : d^{\text{struct}}(P(\lambda)) \le \varepsilon p(|\lambda|)\},$$

where $p(x) = \sum_{k=0}^{n} \alpha_k x^k$.

Theorem 11

Given $\varepsilon > 0$ *and* $P(\lambda) \in M_n^{\text{struct}}(\mathbb{C})$ *a matrix polynomial with* struct $\in$ {Toep, circ, Hankel}, *the* ε*-pseudospectrum and the structured* ε*-pseudospectrum satisfy*

$$\Lambda_\varepsilon^{\text{struct}}(P) = \Lambda_\varepsilon(P).$$

Real structured perturbations

Consider

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n(\mathbb{R})$, $k = 0 : m$ and

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n(\mathbb{R})$. Suppose that $P(\lambda)$ is subject to structured perturbations:

$$[\Delta A_0, \dots, \Delta A_m] = D\Theta[E_0, \dots, E_m],$$

with $D \in M_{n,1}(\mathbb{R})$, $\Theta \in M_{1,t}(\mathbb{R})$ and $E_k \in M_{t,n}(\mathbb{R})$, $k = 0 : m$.
For notational convenience, we introduce

$$E(\lambda) = E[I_n, \lambda I_n, \dots, \lambda^m I_n]^T = \lambda^m E_m + \lambda^{m-1} E_{m-1} + \cdots + E_0,$$

and

$$G(\lambda) = E(\lambda) P(\lambda)^{-1} D = G_R(\lambda) + i G_I(\lambda), \quad G_R(\lambda), G_I(\lambda) \in \mathbb{R}^t.$$

Definition and characterisation of pseudospectra

Definition 11

The structured ε-pseudospectrum is defined by

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbb{C} : (P(\lambda) + D\Theta E(\lambda))x = 0 \text{ for some } x \neq 0, \|\Theta\| \leq \varepsilon\}$$

We denote for $x, y \in \mathbb{R}^t$,

$$d(x, \mathbb{R}y) = \inf_{\alpha\in\mathbb{R}} \|x - \alpha y\|,$$

the distance of the point x from the linear subspace $\mathbb{R}y = \{\alpha y, \alpha \in \mathbb{R}\}$.

Theorem 12

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbb{C} \setminus \Lambda(P) : d(G_R(\lambda), \mathbb{R}G_I(\lambda)) \geq 1/\varepsilon\} \cup \Lambda(P)$$

Conclusion

We have

- The structured pseudospectrum is equal to the pseudospectrum for the two following structures: Toeplitz and circulant
- This result is false for structures Hermitian and skew-Hermitian
- We have generalized these results to pseudospectra of matrix polynomials.
- We have given a formula for structured pseudospectra of real matrix polynomials

Open problems

Pseudozeros of interval polynomials

Problem

Given

- an ball polynomial $p(x) = \sum_{i=0}^{n} B(a_i, r_i)x^i$ with $a_i \in \mathbb{C}$, $r_i \geq 0$ and
- $z \in \mathbb{C}$

does there exist $c_i \in B(a_i, r_i)$ such that $p_c(z) := \sum_{i=0}^{n} c_i z^i = 0$

Solution [Mosier (1986)]

The c_i exist if and only if

$$\frac{|p(z)|}{r_0 + r_1|z| + \cdots + r_n|z|^n} \leq 1$$

Pseudozeros of interval polynomials

Problem

Given

- an interval polynomial $p(x) = \sum_{i=0}^{n} [a_i; b_i] x^i$ with $a_i, b_i \in \mathbb{R}$, $a_i \leq b_i$ and
- $z \in \mathbb{C}$

does there exist $c_i \in [a_i, b_i]$ such that $p_c(z) := \sum_{i=0}^{n} c_i z^i = 0$

Given a vector $d := (d_0, \ldots, d_n)^T$ in $\mathbb{C}^{n+1}$, we consider the weighted norms

$$\|x\|_{\infty,d} = \max_{i=0:n}\{|p_i|/|d_i|\} \quad \text{and} \quad \|x\|_{1,d} := \sum_{i=0}^{n} |d_i||x_i|.$$

Pseudozeros of interval polynomials

We define

$$\mathrm{dist}_d(x,\mathbb{R}y) = \inf_{\alpha\in\mathbb{R}} \|x-\alpha y\|_{1,d},$$

$$\mathrm{dist}_d(x,\mathbb{R}y) = \begin{cases} \min_{\substack{i=0:n \\ y_i\neq 0}} \|x-(x_i/y_i)y\|_{1,d} & \text{if } y\neq 0, \\ \|x\|_{1,d} & \text{if } y=0. \end{cases}$$

and $G_R(p,z)$ and $G_I(p,z)$ being the real and imaginary parts of

$$G(p,z) = \frac{1}{p(z)}(1,z,\ldots,z^n)^T,\ z\in\mathbb{C} \text{ with } p(z)\neq 0.$$

Pseudozeros of interval polynomials

Let $p_m(x) = \sum_{i=0}^{n} m_i x^i$ with $m_i = (a_i + b_i)/2$ et $d_i := (b_i - a_i)/2$.

Solution

The c_i exist if and only if either $p(z) = 0$ or

$$\mathrm{dist}_d(G_R(p_m, z), \mathbb{R}G_I(p_m, z)) \geq 1$$

Pseudozeros of interval polynomials

Problem

Given

- an ball polynomial $p(x) = \sum_{i=0}^{n}([a_j; b_j] + i[c_j; d_j])x^j$ with $a_j, b_j, c_j, d_j \in \mathbb{C}$, $a_j \geq b_j$, $c_j \geq d_j$ and
- $z \in \mathbb{C}$

does there exist $\alpha_j \in [a_j; b_j]$ and $\beta_j \in [c_j; d_j]$ such that $p_c(z) := \sum_{i=0}^{n}(\alpha_j + i\beta_j)z^i = 0$

For the moment, no closed formula ! Maybe NP-hard ?

Thank you for your attention