A note on a nearest polynomial with a given root

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Abstract

In this paper, we consider the problem of a nearest polynomial with a given root in the complex field (the coefficients of the polynomial and the root are complex numbers). We are interested in the existence and the uniqueness of such polynomials. Then we study the problem in the real case (the coefficients of the polynomial and the root are real numbers), and in the real-complex case (the coefficients of the polynomial are real numbers and the root is a complex number). We derive new formulas for computing such polynomials.

Key words: pseudozero set, nearest polynomial, polynomial roots

AMS Subject Classifications: 12D10, 30C10, 30C15, 26C10

1 Introduction

The computation of polynomial roots is extensively used in several fields of Scientific Computing and Engineering. The use of computers implies a round-off of the polynomial coefficients, often due to finite precision (in general using the IEEE-754 norm). The sensitivity of the roots with respect to the uncertainty of the polynomial coefficients has been studied with several approaches.

One of these approaches is to consider the uncertainty of the coefficients (due to round-off) as a continuity problem. This method was first introduced by Ostrowski [10]. The most powerful tool of this method seems to be the *pseudozero set* of a polynomial. Roughly speaking, it is the set of roots of polynomials which are near to a given polynomial. This set was first introduced by Mosier [9]. One may compare this notion with the well-know notion of *pseudospectra*. Concerning pseudospectra, a comparison between the pseudozero set of a polynomial and the pseudospectra of its companion matrix is studied in Trefethen and Toh [14]. A survey on recent results on pseudozero set is given in the book [13].

As noticed in [13], the nearest polynomial with a given root is needed to compute pseudozero set. Besides its relationship to the pseudozero set or to the approximate GCD problem, computing nearest polynomials with given properties has applications in Control Theory [1, 11], as well as other areas if Applied Mathematics. The aim of this paper is to give explicit formulas for such polynomial both with complex polynomials and real polynomials. The case with complex polynomials has already been studied by Hitz and Kaltofen [3, 4, 5] and by Stetter [12]. The main contribution of this note is an explicit formula for the case of real polynomials.

The paper is organized as follows. In Section 2, we state the problem that we will deal with and provide some notations. In Section 3, we recall Stetter's results and prove the uniqueness for particular norms. In Section 4, we give explicit form for such a nearest polynomial. In Section 5, we study the problem in the real case and give an explicit expression of the nearest polynomial for the 2-norm.

2 Preliminaries

Let $\mathcal{P}_n(\mathbf{C})$ be the linear space of polynomials of degree at most n with complex coefficients. For a polynomial $p \in \mathcal{P}_n(\mathbf{C})$ of degree n, we denote by $p_0, \ldots, p_n$ its coefficients, *i.e.*

$$p(z) = \sum_{i=0}^{n} p_i z^i.$$

Given a norm $\|\cdot\|$ on $\mathbf{C}^{n+1}$, the norm on $\mathcal{P}_n(\mathbf{C})$ is defined as the norm on $\mathbf{C}^{n+1}$ of the vector $(p_0, p_1, \ldots, p_n)^T$, *i.e.*

$$\|p\| := \|(p_0, p_1, \ldots, p_n)^T\|.$$

We also define its *dual norm* (denoted $\|\cdot\|_*$) by

$$\|y\|_* = \sup_{\|x\|=1} |y^T x|, \quad y \in \mathbf{C}^{n+1}.$$

This is not the classical definition: we used the transpose rather than the conjugate transpose. Nevertheless, we still have the Hölder inequality: $|y^T x| \leq \|y\|_* \|x\|$. We can even prove that for all y, there exists z (called a *dual vector* of y) satisfying $z^T y = \|z\| \|y\|_* = 1$.

The problem we deal with in this paper is the following one: given a polynomial $p \in \mathcal{P}_n(\mathbf{C})$ and $u \in \mathbf{C}$, we are looking for a nearest polynomial $p_u \in \mathcal{P}_n(\mathbf{C})$ to p having the root u. This problem appears naturally in algorithms for drawing pseudozero set (see [9, 13, 14, 15]).

Let $p \in \mathcal{P}_n(\mathbf{C})$ be a polynomial of degree n and $u \in \mathbf{C}$ be a complex number that will be a root of the polynomial we are looking for. The problem can be formulated as follow:

> *Find a polynomial $p_u \in \mathcal{P}_n(\mathbf{C})$ satisfying $p_u(u) = 0$ and such that if there exists a polynomial $q \in \mathcal{P}_n(\mathbf{C})$ with $q(u) = 0$ then we have $\|p - p_u\| \leq \|p - q\|$.*

Such a problem was first introduced by Hitz and Kaltofen [3, 4, 5] using the 2-norm with complex polynomials. Their results were generalized by Stetter [12, 13] for Hölder p-norm ($1 \leq p \leq \infty$) still using complex polynomials. In this paper, we establish the version of Stetter's results (in a slightly different way), and in addition, we study the uniqueness of a nearest polynomial. We also establish a computable formula for a nearest polynomial in the real-complex case. We give an explicit formula for such a polynomial in the case of the 2-norm.

3 The complex case

Let us denote $\underline{u} := (1, u, u^2, \dots, u^n)^T$. It is well known (see [6, p. 278]) there exists a vector $d := (d_0, d_1, \dots, d_n)^T \in \mathbf{C}^{n+1}$ satisfying $d^T \underline{u} = \|\underline{u}\|_*$ and $\|d\| = 1$. Let us define the polynomials r and p_u by

$$\begin{aligned} r(z) &= \sum_{k=0}^{n} r_k z^k \quad \text{with} \quad r_k = d_k, \\ p_u(z) &= p(z) - \frac{p(u)}{r(u)} r(z). \end{aligned}$$

We can prove the following theorem.

Theorem 1. *The polynomial p_u is a nearest polynomial to p with the root u.*

Proof. It is clear that $p_u(u) = 0$. We have to prove that p_u is a nearest polynomial to p satisfying this equality. Let $q \in \mathcal{P}_n(\mathbf{C})$ verifying $q(u) = 0$. By applying Hölder inequality, we have

$$|p(z) - q(z)| = \left| \sum (p_i - q_i) z^i \right| \leq \|p - q\| \|\underline{z}\|_*.$$

Letting $z = u$ in the previous inequality yields

$$|p(u)| \leq \|p - q\| \|\underline{u}\|_* \quad \text{and so} \quad \frac{|p(u)|}{\|\underline{u}\|_*} \leq \|p - q\|. \tag{3.1}$$

Furthermore, we remark that

$$\|p - p_u\| = \frac{|p(u)|}{|r(u)|} \|r\|.$$

As $\|r\| = \|d\| = 1$ and $r(u) = d^T \underline{u} = \|\underline{u}\|_*$, we have

$$\|p - p_u\| = \frac{|p(u)|}{\|\underline{u}\|_*}.$$

It follows from (3.1) that

$$\|p - p_u\| \leq \|p - q\|.$$

In the following proposition, we have uniqueness of a nearest polynomial if the norm is strictly convex (it is the case for the p-norms $\|\cdot\|_p$, $1 < p < \infty$). It is not true for an arbitrary norm.

Proposition 1. *If the norm $\|\cdot\|$ is strictly convex then the nearest polynomial is unique.*

Proof. Let p_u and p'_u be two distinct nearest polynomials to p having the root u. Let us denote $d := \|p - p_u\| = \|p - p'_u\|$. We have

$$\begin{aligned} \|p - (p_u + p'_u)/2\| &= \|(p - p_u)/2 + (p - p'_u)/2\| \\ &< \frac{1}{2}(\|p - p_u\| + \|p - p'_u\|) \\ &< d. \end{aligned}$$

This contradicts the property of p_u and p'_u to be a nearest polynomial. So, we have $p_u = p'_u$.

Remark 1. The non-uniqueness of a nearest polynomial for the norm $\|\cdot\|_1$ and $\|\cdot\|_\infty$ proceeds from the particular form of the unity ball. We have the following counter-examples:

- in the case of 1-norm $\|\cdot\|_1$, if we take $p(z) = 1 + z$ and $u = 1$ then the two polynomials $p_1^{(1)}(z) = 0$ and $p_1^{(2)}(z) = \frac{1}{3}(1 - z)$ are both nearest polynomials;
- in the case of the ∞-norm $\|\cdot\|_\infty$, if we take $p(z) = 1 + z$ and $u = 0$ then the two polynomials $p_0^{(1)}(z) = z$ et $p_0^{(2)}(z) = \frac{1}{2}z$ are both nearest polynomials.

4 Computation of p_u

In this section, we are concerned with finding an explicit expression for the polynomial p_u. In the previous section, we have established the general formula for p_u, but we have to compute this polynomial, that is to say, find a dual vector of $\underline{u}$. The problem can be formulated as follow:

Find $d \in \mathbf{C}^{n+1}$ *satisfying* $d^T\underline{u} = \|\underline{u}\|_*$ *and* $\|d\| = 1$.

This problem has no explicit solution for an arbitrary norm. We are going to compute it for Hölder p-norm. We will denote $u = |u|e^{i\theta}$ for $u \neq 0$.

4.1 For the ∞-norm

The problem is the following one.

Find $d \in \mathbf{C}^{n+1}$ *satisfying*

$$\sum_{j=0}^{n} d_j u^j = \sum_{j=0}^{n} |u|^j \quad \textit{and} \quad \max_{j=1,\ldots,n} |d_j| = 1.$$

We remark that if $u \neq 0$ then $d_j = e^{-ij\theta}$ is convenient. In this case, p_u can be written

$$p_u(z) = p(z) - \frac{p(u)}{\sum_{j=0}^{n} |u|^j} \sum_{j=0}^{n} e^{-ij\theta} z^j.$$

Otherwise, if $u = 0$, then $d = (1, 0, \ldots, 0)$ is convenient and in this case, p_u can be written

$$p_u(z) = p(z) - p(u).$$

Example 1. Let us take an easy example: $p(z) = z + 1$ et $u = 1/2$. In this case $\underline{u} = (1, 1/2)$ and $r(z) = z+1$. By applying the previous work, we obtain $p_u(z) = 1+z-((1+1/2)/(1+1/2))(1+z) = 0$. Let us justify this result. Indeed, p_u is in the form $\alpha(z - 1/2)$, $\alpha \in \mathbf{C}$. In this case

$$\|p - \alpha(z - 1/2)\|_\infty = \max\{|1 - \alpha|, |1 + (1/2)\alpha|\} =: d.$$

Three cases arise

- if $\alpha > 0$ then $d = 1 + (1/2)\alpha > 1$;
- if $\alpha < 0$ then $d = 1 - \alpha > 1$;
- if $\alpha = 0$ then $d = 1$.

The minimum is obtain for $\alpha = 0$.

4.2 For the p-norm ($1 \leq p < \infty$)

The problem can be expressed as follow.

Find $d \in \mathbf{C}^{n+1}$ satisfying

$$\sum_{j=0}^{n} d_j u^j = \|\underline{u}\|_q \quad \textit{and} \quad \|d\|_p = 1 \quad \textit{with} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

It is like solving with $d = (d_j)$ the following equations:

$$\sum_{j=0}^{n} d_j u^j = \Big[\sum_{j=0}^{n} |u^j|^q\Big]^{1/q}, \tag{4.2}$$

$$\Big[\sum_{j=0}^{n} |d_j|^p\Big]^{1/p} = 1. \tag{4.3}$$

If $u \neq 0$, let us take $d_j := \frac{|u^j|^{q-1}e^{-ij\theta}}{\|\underline{u}\|_q^{q-1}}$. Let us verify that $d = (d_j)$ is convenient. It is clear that d satisfies equation (4.2). Equation (4.3) yields

$$\begin{aligned}\Big[\sum_{j=0}^{n} |d_j|^p\Big]^{1/p} &= \frac{1}{\|\underline{u}\|_q^{q-1}}\Big[\sum_{j=0}^{n} |u^j|^{p(q-1)}\Big]^{1/p} = \frac{1}{\|\underline{u}\|_q^{q-1}}\Big[\sum_{j=0}^{n} |u^j|^{q}\Big]^{1/p}, \\ &= \frac{1}{\|\underline{u}\|_q^{q-1}}\Big[\|\underline{u}\|_q^q\Big]^{1/p} = \frac{\|\underline{u}\|_q^{q/p}}{\|\underline{u}\|_q^{q-1}} = \|\underline{u}\|_q^{q/p-q+1} = \|\underline{u}\|_q^0 = 1.\end{aligned}$$

So the vector d is a solution of our problem. The polynomial p_u is

$$p_u(z) = p(z) - \frac{p(u)}{\sum_{j=0}^{n} |u^j|^q} \sum_{j=0}^{n} |u^j|^{q-1} e^{-ij\theta} z^j.$$

If now $u = 0$ then $d = (1, 0, \ldots, 0)$ is convenient and the polynomial p_u is

$$p_u(z) = p(z) - p(u).$$

5 The real case

In this section, we show that following a method proposed in [2], we can solve the problem of a nearest polynomial in the real case.

Let $\mathcal{P}_n(\mathbf{R})$ be the linear space of polynomials of degree at most n with real coefficients. Let $p \in \mathcal{P}_n(\mathbf{R})$ be given by

$$p(z) = \sum_{i=0}^{n} p_i z^i.$$

Representing p by the vector $(p_0, \ldots, p_{n-1}, p_n)$ of its coefficients, we identify the norm $\|\cdot\|$ on $\mathcal{P}_n(\mathbf{R})$ to the norm on $\mathbf{R}^{n+1}$ of the corresponding vector.

Let $p \in \mathcal{P}_n(\mathbf{R})$ and $u \in \mathbf{R}$ a real number which will be a root of the polynomial we are looking for. The problem is the following one:

> *Find a polynomial $p_u \in \mathcal{P}_n(\mathbf{R})$ satisfying $p_u(u) = 0$ and such that if there exists a polynomial $q \in \mathcal{P}_n(\mathbf{R})$ with $q(u) = 0$ then we have $\|p - p_u\| \leq \|p - q\|$.*

If we apply the same procedure as in the complex case, we obtain a solution.

A more complicated problem appears when u is a complex number. Let $p \in \mathcal{P}_n(\mathbf{R})$ be a polynomial of degree n and $u \in \mathbf{C}$ be a complex number which will be a root of the polynomial we are looking for. We deal with the following problem:

> *Find a polynomial $p_u \in \mathcal{P}_n(\mathbf{R})$ satisfying $p_u(u) = 0$ and such that if there exists a polynomial $q \in \mathcal{P}_n(\mathbf{R})$ with $q(u) = 0$ then we have $\|p - p_u\| \leq \|p - q\|$.*

For solving this problem, we used arguments proposed in [2]. We suppose that $p(u) \neq 0$, otherwise we can choose $p_u = p$. Let $q \in \mathcal{P}_n(\mathbf{R})$ such that $q(u) = 0$. We have $p(u) = p(u) - q(u) = (p - q)^T \underline{u}$ where $\underline{u} = (1, u, u^2, \ldots, u^n)$. It follows that $1 = (p - q)^T G(u)$ where $G(u) = \frac{1}{p(u)}(1, u, \ldots, u^n)^T$. Let us denote $G(u) = G_R(u) + iG_I(u)$ where $G_R(u)$, $G_I(u)$ are the real and imaginary parts of $G(u)$. It follows that $1 = (p - q)^T G_R(u) + i(p - q)^T G_I(u)$. As a consequence, we have

$$\begin{cases} (p - q)^T G_R(u) = 1, \\ (p - q)^T G_I(u) = 0. \end{cases}$$

So we have $\|p - q\| \|G_R(u) - \alpha G_I(u)\|_* \geq 1$, for all $\alpha \in \mathbf{R}$. We denote for $x, y \in \mathbf{R}^{n+1}$,

$$d(x, \mathbf{R}y) = \inf_{\alpha \in \mathbf{R}} \|x - \alpha y\|_*,$$

the distance of a point $x \in \mathbf{R}^{n+1}$ from the linear subspace $\mathbf{R}y = \{\alpha y, \alpha \in \mathbf{R}\}$. We can conclude that $\|p - q\| \geq \frac{1}{d(G_R(u), \mathbf{R}G_I(u))}$. From a duality theorem (see [8, p.119]), there exists a vector $z \in \mathbf{R}^{n+1}$ with $\|z\| = 1$ satisfying

$$z^T G_R(u) = d(G_R(u), \mathbf{R}G_I(u)) \quad \text{and} \quad z^T G_I(u) = 0.$$

Let us consider the polynomial with coefficients $p_u = p - \frac{z}{d(G_R(u), \mathbf{R}G_I(u))}$.

Theorem 2. *The polynomial p_u is a nearest polynomial with real coefficients having root u.*

Proof. We have

$$p_u(u) = p(u) - \frac{z^T \underline{u}}{d(G_R(u), \mathbf{R}G_I(u))} = p(u) - \frac{p(u) z^T G(u)}{d(G_R(u), \mathbf{R}G_I(u))} = 0$$

and

$$\|p - p_u\| = \left\| \frac{z}{d(G_R(u), \mathbf{R}G_I(u))} \right\| = \frac{1}{d(G_R(u), \mathbf{R}G_I(u))}.$$

It follows that p_u is a nearest real polynomial to p with root u.

The main difficulty for computing p_u is the computation of $d(G_R(u), \mathbf{R}G_I(u))$ and z. These quantities can be calculated easily for the 2-norm. Let us now denote the 2-norm $\| \cdot \|_2$ and $\langle \cdot, \cdot \rangle$ the corresponding inner product. In this case, we have

$$d(x, \mathbf{R}y) = \begin{cases} \sqrt{\|x\|_2^2 - \frac{\langle x, y \rangle^2}{\|y\|_2^2}} & \text{if } y \neq 0, \\ \|x\|_2 & \text{if } y = 0. \end{cases}$$

Moreover, a vector z satisfying

$$\langle z, x\rangle = d(x, \mathbf{R}y), \quad \langle z, y\rangle = 0 \quad \text{and} \quad \|z\|_2 = 1$$

is given by

$$z = \frac{x - \langle x, y\rangle \frac{y}{\|y\|_2^2}}{\left\| x - \langle x, y\rangle \frac{y}{\|y\|_2^2} \right\|_2} \quad \text{if} \quad y \neq 0,$$

and

$$z = \frac{x}{\|x\|_2} \quad \text{if} \quad y = 0.$$

For the ∞-norm, it is shown in [7, Prop. 7.7.2] that

$$d(x, \mathbf{R}y) = \begin{cases} \min\limits_{\substack{i=0:n \\ y_i \neq 0}} \|x - (x_i/y_i)y\|_1 & \text{if } y \neq 0, \\ \|x\|_1 & \text{if } y = 0. \end{cases}$$

We need to find a vector z satisfying

$$\langle z, x\rangle = d(x, \mathbf{R}y), \quad \langle z, y\rangle = 0 \quad \text{and} \quad \|z\|_\infty = 1.$$

This is a difficult task and there is no easy computable formula for this problem. For the other p-norm with $p \neq 2, \infty$, there is no easy computable formula to calculate $d(x, \mathbf{R}y)$.

6 Conclusion

In the paper, we give a formula for the problem of a nearest polynomial with a given root for the three following cases:

- the polynomial has complex coefficients and the root is complex;
- the polynomial has real coefficients and the root is real;
- the polynomial has real coefficients and the root is complex.

Only the first and the second cases yields an explicit expression for the polynomial (for the p-norm). The third case is more difficult. It yields a computational expression but an explicit expression is only derived in the case of the 2-norm.

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