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A note on structured pseudospectra

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Abstract

In this note, we study the notion of structured pseudospectra. We prove that for Toeplitz, circulant, Hankel and symmetric structures, the structured pseudospectrum equals the unstructured pseudospectrum. We show that this is false for Hermitian and skew-Hermitian structures. We generalize the result to pseudospectra of matrix polynomials. Indeed, we prove that the structured pseudospectrum equals the unstructured pseudospectrum for matrix polynomials with Toeplitz, circulant, Hankel and symmetric structures. We conclude by giving a formula for structured pseudospectra of real matrix polynomials. The particular type of perturbations used for these pseudospectra arise in control theory.

MSC: 65F15

Keywords: Structured perturbation; Pseudospectrum; Polynomial matrix; Toeplitz matrix; Circulant matrix; Hankel matrix; Symmetric matrix

1. Introduction and notation

The ε-pseudospectrum of a matrix A has been introduced in [12] as the subset of complex numbers consisting of all eigenvalues of all complex matrices within a distance ε of A. If the matrix A has a certain structure (for example, Toeplitz), it is natural to allow only perturbed matrices with the same structure.

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In this case, the structured ε-pseudospectrum of a structured matrix A is the subset of complex numbers consisting of all eigenvalues of all complex structured matrices within a distance ε of A.

In this paper, we are mainly concerned with the linear structures,

$$\text{struct} \in \{\text{Toep}, \text{circ}, \text{Hankel}, \text{sym}\} \tag{1.1}$$

corresponding to the sets of Toeplitz, circulant, Hankel and symmetric matrices.

Throughout the paper, we denote by $M_n(\mathbf{C})$ the set of complex $n \times n$ matrices and by $M_n^{\text{struct}}(\mathbf{C})$ the set of structured complex matrices, struct as in (1.1). We endow these spaces with the 2-norm (also called the spectral norm) denoted by $\|\cdot\|$.

Let us consider a matrix $A \in M_n(\mathbf{C})$. We denote its spectrum by $\Lambda(A)$. For a real $\varepsilon > 0$, the ε-pseudospectrum of a matrix $A \in M_n(\mathbf{C})$ is the set $\Lambda_\varepsilon(A)$ defined by

$$\Lambda_\varepsilon(A) = \{z \in \mathbf{C} : z \in \Lambda(X) \text{ where } X \in M_n(\mathbf{C}) \text{ and } \|X - A\| \leqslant \varepsilon\}.$$

Given a matrix $A \in M_n^{\text{struct}}(\mathbf{C})$ with struct as in (1.1), the structured ε-pseudospectrum of A is the set $\Lambda_\varepsilon^{\text{struct}}(A)$ defined by

$$\Lambda_\varepsilon^{\text{struct}}(A) = \{z \in \mathbf{C} : z \in \Lambda(X) \text{ where } X \in M_n^{\text{struct}}(\mathbf{C}) \text{ and } \|X - A\| \leqslant \varepsilon\}.$$

For $A \in M_n^{\text{struct}}(\mathbf{C})$, it is clear that we always have

$$\Lambda_\varepsilon^{\text{struct}}(A) \subseteq \Lambda_\varepsilon(A).$$

We are interested in the structures for which there is equality.

To our knowledge, structured pseudospectra (also called "spectral value sets") have been first defined and studied with perturbations of the form

$$A \rightsquigarrow A + \Delta A = A + D\Theta E, \quad \Theta \in M_{l,q}(\mathbf{C}),$$

where $D \in M_{n,l}(\mathbf{C})$, $E \in M_{q,n}(\mathbf{C})$ are fixed matrices defining the structure of the perturbation (see [5,11,1]). The definition of structured pseudospectra, we use in this note was first introduced by Böttcher et al. [3] for the Toeplitz structure. They called it "Toeplitz" ε-pseudospectrum in [3] and Toeplitz-structured pseudospectrum in [2]. In [3], they considered banded Toeplitz matrices only and hence restricted themselves to defining $\Lambda_\varepsilon^{\text{Toep}[r,s]}(A)$ for $A \in M_n^{\text{Toep}[r,s]}(\mathbf{C})$ where Toep$[r, s]$ stands for the Toeplitz matrices with at most r nonzero superdiagonals and at most s nonzero subdiagonals. They established that $\Lambda_\varepsilon(A)$ may be different from $\Lambda_\varepsilon^{\text{Toep}[r,s]}(A)$. In this note, we show equality for $r = s = n$. Moreover, we extend the definition to other structures, such as circulant, Hankel or symmetric structures.

The paper is organized as follows. In Section 2, we recall results on the structured distance to singularity. In Section 3, we prove that for struct $\in$ {Toep, circ, Hankel, sym}, the structured pseudospectrum equals the unstructured pseudospectrum. Then, we study the cases of the Hermitian and skew-Hermitian structures. We prove that the equality of the structured and unstructured pseudospectrum does not hold for these structures. In Section 4, we generalize the previous results to pseudospectra of matrix polynomials with struct $\in$ {Toep, circ, Hankel, sym}. We also consider structured pseudospectra of real matrix polynomials.

2. Results on the structured distance to singularity

In this section, we recall some results on structured distance to singularity. Given a nonsingular matrix $A \in M_n(\mathbf{C})$, we define the distance to singularity by

$$d(A) = \min\{\|\Delta A\| : A + \Delta A \text{ singular}, \Delta A \in M_n(\mathbf{C})\}. \tag{2.2}$$

For a nonsingular matrix $A \in M_n^{\text{struct}}(\mathbf{C})$, we define the structured distance to singularity by

$$d^{\text{struct}}(A) = \min\{\|\Delta A\| : A + \Delta A \text{ singular}, \Delta A \in M_n^{\text{struct}}(\mathbf{C})\}. \tag{2.3}$$

Rump has proved in [9, Theorem 12.2] that the two distances $d(A)$ and $d^{\text{struct}}(A)$ are equal for struct $\in$ {Toep, circ, Hankel}.

Theorem 2.1 (*Rump [9, Theorem 12.2]*). *Let a nonsingular $A \in M_n^{\text{struct}}(\mathbf{C})$ be given for* struct $\in$ {Toep, circ, Hankel}. *Then we have*

$$d(A) = d^{\text{struct}}(A) = \|A^{-1}\|^{-1} = \sigma_{\min}(A).$$

Here, $\sigma_{\min}(A)$ denotes the smallest singular value of A. The same property occurs for the symmetric structure. Before stating the result, we will need the following lemma.

Lemma 2.2 (*Rump [9, Lemma 10.1]*). *Let $x \in \mathbf{C}^n$ be given. Then there exists a complex symmetric matrix A such that $Ax = \overline{x}$ and $\|A\| = 1$.*

The next result can be found in [10]. For the sake of completeness, we recall the proof.

Theorem 2.3 (*Tisseur and Graillat [10]*). *Let A be a nonsingular matrix in $M_n^{\text{struct}}(\mathbf{C})$ where* struct=sym. *Then*

$$d(A) = d^{\text{struct}}(A) = \|A^{-1}\|^{-1} = \sigma_{\min}(A).$$

Proof. Obviously, we have $d^{\text{struct}}(A) \geqslant d(A) = \|A^{-1}\|^{-1} = \sigma_{\min}(A)$, and hence it remains to be shown that $(A + \Delta A)x = 0$ for some $x \neq 0$ and ΔA symmetric with $\|\Delta A\| = \sigma_{\min}(A)$. Let $A = U\Sigma U^{\mathrm{T}}$ be the Takagi's factorization of A where U is unitary and Σ is diagonal with positive entries (Horn and Johnson [7, Corollary 4.4.4]). Let x be the column of U corresponding to the smallest diagonal entry in Σ. Then $A\overline{x} = \sigma_{\min}(A)x$. By Lemma 2.2 there exists a symmetric matrix C such that $C\overline{x} = x$ and $\|C\| = 1$. Let $\Delta A = -\sigma_{\min}(A)C$. Then ΔA is symmetric, $\|\Delta A\| = \sigma_{\min}(A)$ and

$$(A + \Delta A)\overline{x} = \sigma_{\min}(A)x - \sigma_{\min}(A)x = 0$$

so that $A + \Delta A$ is singular. $\square$

3. Structured pseudospectrum equals unstructured pseudospectrum

The following lemma shows that the ε-pseudospectrum is linked to the distance to singularity. This is a well-known result (see [13]).

Lemma 3.1. *Given* $\varepsilon > 0$ *and* $A \in M_n(\mathbf{C})$, *the* ε*-pseudospectrum satisfies*

$$\Lambda_\varepsilon(A) = \{z \in \mathbf{C} : d(A - zI) \leqslant \varepsilon\}.$$

In this section, we deal with

$$\text{struct} \in \{\text{Toep, circ, sym}\} \tag{3.4}$$

As we have seen before, we have $d(A) = d^{\text{struct}}(A)$ for $A \in M_n^{\text{struct}}(\mathbf{C})$. Hence, it is sufficient to prove that

$$\Lambda_\varepsilon^{\text{struct}}(A) = \{z \in \mathbf{C} : d^{\text{struct}}(A - zI) \leqslant \varepsilon\}$$

in order to conclude that $\Lambda_\varepsilon(A) = \Lambda_\varepsilon^{\text{struct}}(A)$ for a given matrix $A \in M_n^{\text{struct}}(\mathbf{C})$. This is the aim of the following lemma.

Lemma 3.2. *Given* $\varepsilon > 0$ *and* $A \in M_n^{\text{struct}}(\mathbf{C})$ *with* struct *as in* (3.4), *the structured* ε*-pseudospectrum satisfies*

$$\Lambda_\varepsilon^{\text{struct}}(A) = \{z \in \mathbf{C} : d^{\text{struct}}(A - zI) \leqslant \varepsilon\}.$$

The proof is very similar to the one of Lemma 3.1 but we have to pay attention to keep the structure.

Proof. With A also zI and $A - zI$ is in $M_n^{\text{struct}}(\mathbf{C})$, so

$$\begin{aligned} z \in \Lambda_\varepsilon^{\text{struct}}(A) &\Leftrightarrow \exists \Delta A \in M_n^{\text{struct}}(\mathbf{C}) : \det(A - zI + \Delta A) = 0, \quad \|\Delta A\| \leqslant \varepsilon \\ &\Leftrightarrow d^{\text{struct}}(A - zI) \leqslant \varepsilon. \qquad \square \end{aligned}$$

From Lemmas 3.1 and 3.2 and Theorems 2.1 and 2.3, we deduce the following theorem.

Theorem 3.3. *Given* $\varepsilon > 0$ *and* $A \in M_n^{\text{struct}}(\mathbf{C})$ *with* struct $\in$ {Toep, circ, sym}, *the* ε*-pseudospectrum and the structured* ε*-pseudospectrum satisfy*

$$\Lambda_\varepsilon^{\text{struct}}(A) = \Lambda_\varepsilon(A).$$

Theorem 2.1 is also true for the Hermitian and skew-Hermitian structures. However, the proof of Lemma 3.2 given above does not work for these two structures (and also not for the Hankel structure) since the scalar matrices (zI for $z \in \mathbf{C}$) do not have these structures.

In fact, we do not have equality between the structured and the unstructured pseudospectrum for the Hermitian and skew-Hermitian structures. Indeed, Hermitian and skew-Hermitian matrices are normal, and if $A \in M_n(\mathbf{C})$ is normal then $\Lambda_\varepsilon(A) = \{z \in \mathbf{C} : \operatorname{dist}(z, \Lambda(A)) \leqslant \varepsilon\}$ (see [12]). Consequently, $\Lambda_\varepsilon(A)$ contains an open subset of $\mathbf{C}$. But if A is Hermitian then obviously $\Lambda_\varepsilon^{\text{herm}}(A) \subset \mathbf{R}$, while if A is skew-Hermitian it is easily seen that $\Lambda_\varepsilon^{\text{skewherm}}(A) \subset \mathrm{i}\mathbf{R}$. This shows that for Hermitian and skew-Hermitian matrices A the pseudospectrum is always strictly larger than the structured pseudospectrum. It is clear that $\Lambda_\varepsilon^{\text{herm}}(A) \subset \Lambda_\varepsilon(A) \cap \mathbf{R}$. Let $z \in \Lambda_\varepsilon(A) \cap \mathbf{R}$. Since now zI is Hermitian, it follows that $d^{\text{herm}}(A - zI) = d(A - zI)$ so $z \in \Lambda_\varepsilon^{\text{herm}}(A)$. Consequently, $\Lambda_\varepsilon^{\text{herm}}(A) = \Lambda_\varepsilon(A) \cap \mathbf{R}$. With the same arguments, we conclude that $\Lambda_\varepsilon^{\text{skewherm}}(A) = \Lambda_\varepsilon(A) \cap \mathrm{i}\mathbf{R}$.

As observed by the referees, the equality $\Lambda_\varepsilon^{\mathrm{Hankel}}(A) = \Lambda_\varepsilon(A)$ is nevertheless true for matrices A in $M_n^{\mathrm{Hankel}}(\mathbf{C})$. To see this, let $z \in \Lambda_\varepsilon(A)$. As in the proof of Theorem 2.3, consider the Takagi's factorization $A-zI=U\Sigma U^{\mathrm{T}}$ and take an $x \neq 0$ such that $(A-zI)\overline{x}=\sigma_{\min}(A-zI)x$. Rump [9] showed that the matrix A in Lemma 2.2 can actually be chosen as a Hankel matrix. Thus, there is a matrix $C \in M_n^{\mathrm{Hankel}}(\mathbf{C})$ such that $C\overline{x}=x$ and $\|C\|=1$. It follows that $\Delta A := -\sigma_{\min}(A-zI)C$ is a Hankel matrix and that $(A-zI+\Delta A)\overline{x}=0$. Consequently, $z \in \Lambda_\varepsilon^{\mathrm{Hankel}}(A)$.

4. Structured pseudospectra of matrix polynomials

This section deals with pseudospectra of matrix polynomials (see [4,8,11]). We prove a result analogous to Theorem 3.3 for the pseudospectra of matrix polynomials in the first subsection. The second subsection is concerned with structured pseudospectra of real matrix polynomials taking into account only real perturbations.

4.1. Structured pseudospectra of complex matrices

The polynomial eigenvalue problem is to find the solutions $(x, \lambda) \in \mathbf{C}^n \times \mathbf{C}$ of

$$P(\lambda)x = 0,$$

where

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n(\mathbf{C})$, $k=0:m$. If $x \neq 0$ then λ is called an eigenvalue and x the corresponding eigenvector. The set of eigenvalues of P is denoted $\Lambda(P)$. When A_m is nonsingular, P is said to be *regular* and has mn eigenvalues. In the sequel, we assume that P is regular. Let us define

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n(\mathbf{C})$. We define the ε-pseudospectrum of P by

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbf{C} : (P(\lambda) + \Delta P(\lambda))x = 0 \text{ for some } x \neq 0 \text{ with } \|\Delta A_k\| \leqslant \alpha_k \varepsilon,\ k = 0 : m\}.$$

The nonnegative parameters $\alpha_1, \ldots, \alpha_m$ allow freedom in how perturbations are measured. In the previous definition, we also assume that all the matrix polynomials $P(\lambda) + \Delta P(\lambda)$ are also regular. The following lemma is a reformulation of Lemma 2.1 in [11].

Lemma 4.1. *We have*

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbf{C} : d(P(\lambda)) \leqslant \varepsilon p(|\lambda|)\},$$

where $p(x) = \sum_{k=0}^{m} \alpha_k x^k$.

Proof. Let λ be in $\Lambda_\varepsilon(P)$. This implies that there exists $\Delta P(\lambda) \in M_n(\mathbf{C})$ such that $\|\Delta A_k\| \leqslant \alpha_k \varepsilon$, $k=0:m$ and $P(\lambda) + \Delta P(\lambda)$ is singular. It follows from the definition of the distance d that $d(P(\lambda)) \leqslant \|\Delta P(\lambda)\|$.

Since

$$\|\Delta P(\lambda)\| \leqslant \sum_{k=0}^{m} |\lambda|^k \alpha_k \varepsilon = \varepsilon p(|\lambda|),$$

we have $d(P(\lambda)) \leqslant \varepsilon p(|\lambda|)$.

Conversely, let $\lambda \in \mathbf{C}$ be such that $d(P(\lambda)) \leqslant \varepsilon p(|\lambda|)$. This means that there exists $X \in M_n(\mathbf{C})$ such that $\|X\| \leqslant \varepsilon p(|\lambda|)$ and $P(\lambda) + X$ is singular. Let us define ΔA_k by

$$\Delta A_k = \mathrm{sign}(\lambda^k)\alpha_k p(|\lambda|)^{-1} X,$$

where for complex z we define

$$\mathrm{sign}(z) = \begin{cases} |z|/z, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Then

$$\Delta P(\lambda) = \sum_{k=0}^{m} \lambda^k \Delta A_k = \left(\sum_{k=0}^{m} |\lambda|^k \alpha_k p(|\lambda|)^{-1} X \right) = X$$

and $\|\Delta A_k\| \leqslant \alpha_k \varepsilon$, $k = 0 : m$. Hence $\lambda \in \Lambda_\varepsilon(P)$. □

We assume now that the matrices A_k have a certain structure belonging to

$$\text{struct} \in \{\text{Toep, circ, Hankel, sym}\}. \tag{4.5}$$

We also suppose that all the matrices A_k and ΔA_k, $k = 0 : n$, belong to $M_n^{\text{struct}}(\mathbf{C})$ for a given structure in (4.5). Let

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n^{\text{struct}}(\mathbf{C})$, $k = 0 : m$ and

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n^{\text{struct}}(\mathbf{C})$. One notices that $P(\lambda)$ and $\Delta P(\lambda)$ belong to $M_n^{\text{struct}}(\mathbf{C})$. We define the structured ε-pseudospectrum of P by

$$\begin{aligned}\Lambda_\varepsilon^{\text{struct}}(P) = \{\lambda \in \mathbf{C} : (P(\lambda) + \Delta P(\lambda))x = 0 \text{ for some } x \neq 0 \\ \text{with } \Delta A_k \in M_n^{\text{struct}}(\mathbf{C}),\ \|\Delta A_k\| \leqslant \alpha_k \varepsilon,\ k = 0 : n\}.\end{aligned}$$

The following lemma is the structured version of Lemma 4.1.

Lemma 4.2. *For struct as in* (4.5) *we have*

$$\Lambda_\varepsilon^{\text{struct}}(P) = \{\lambda \in \mathbf{C} : d^{\text{struct}}(P(\lambda)) \leqslant \varepsilon p(|\lambda|)\},$$

where $p(x) = \sum_{k=0}^{n} \alpha_k x^k$.

Proof. The proof is almost identical to the one of Lemma 4.1. The main thing to notice is that the matrix X and so the matrices ΔA_k defined in the proof of Lemma 4.1 can be chosen in $M_n^{\text{struct}}(\mathbf{C})$. $\square$

From Lemmas 4.1 and 4.2 and Theorems 2.1 and 2.3 we deduce the following theorem for struct in (4.5).

Theorem 4.3. *If* $\varepsilon > 0$ *and* $P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0$ *is a matrix polynomial with* $A_k \in M_n^{\text{struct}}(\mathbf{C})$, $k = 0 : m$ *and*

$$\text{struct} \in \{\text{Toep, circ, Hankel, sym}\},$$

then the ε*-pseudospectrum and the structured* ε*-pseudospectrum satisfy*

$$\Lambda_\varepsilon^{\text{struct}}(P) = \Lambda_\varepsilon(P).$$

4.2. Structured pseudospectra of real matrix polynomials

In this subsection, we consider

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

with $A_k \in M_n(\mathbf{R})$, $k = 0 : m$ and

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \cdots + \Delta A_0,$$

where $\Delta A_k \in M_n(\mathbf{R})$. We suppose that $P(\lambda)$ is subject to structured perturbations that can be expressed as

$$[\Delta A_0, \ldots, \Delta A_m] = D\Theta[E_0, \ldots, E_m],$$

with $D \in M_{n,1}(\mathbf{R})$, $\Theta \in M_{1,t}(\mathbf{R})$ and $E_k \in M_{t,n}(\mathbf{R})$, $k = 0 : m$. This type of structure arises naturally in control theory. For notational convenience, we introduce

$$E(\lambda) = E[I_n, \lambda I_n, \ldots, \lambda^m I_n]^{\mathrm{T}} = \lambda^m E_m + \lambda^{m-1} E_{m-1} + \cdots + E_0,$$

and

$$G(\lambda) = E(\lambda) P(\lambda)^{-1} D = G_R(\lambda) + \mathrm{i} G_I(\lambda), \quad G_R(\lambda), G_I(\lambda) \in \mathbf{R}^t.$$

We define the structured ε-pseudospectrum by

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbf{C} : (P(\lambda) + D\Theta E(\lambda))x = 0 \text{ for some } x \neq 0, \|\Theta\| \leqslant \varepsilon\}.$$

We assume that the matrix polynomial P as well as all the matrix polynomials $P(\lambda) + D\Theta E(\lambda)$ are regular. For $x, y \in \mathbf{R}^t$, we denote by

$$d(x, \mathbf{R}y) = \inf_{\alpha \in \mathbf{R}} \|x - \alpha y\|,$$

the distance of the point x from the linear subspace $\mathbf{R}y = \{\alpha y, \alpha \in \mathbf{R}\}$. The following theorem provides a computable characterization of the structured pseudospectrum.

Theorem 4.4. *We have*

$$\Lambda_\varepsilon(P) = \{\lambda \in \mathbf{C}\backslash\Lambda(P) : d(G_R(\lambda), \mathbf{R}G_I(\lambda)) \geqslant 1/\varepsilon\} \cup \Lambda(P).$$

Proof. If there exists $x \neq 0$ such that $(P(\lambda) + D\Theta E(\lambda))x = 0$ then $x = -P(\lambda)^{-1}D\Theta E(\lambda)x$ so that $\Theta E(\lambda)x = -\Theta E(\lambda)P(\lambda)^{-1}D\Theta E(\lambda)x$. Let us write $u = \Theta E(\lambda)x \in \mathbf{C}$, $u = u_1 + iu_2$, $(u_1, u_2) \in \mathbf{R}^2$. It is clear that $u \neq 0$ since $\lambda \notin \Lambda(P)$. Using these notations, we obtain

$$u = -\Theta G(\lambda)u.$$

This can be rewritten in real terms by

$$\begin{aligned} u_1 &= -\Theta G_R(\lambda)u_1 + \Theta G_I(\lambda)u_2, \\ u_2 &= -\Theta G_R(\lambda)u_2 - \Theta G_I(\lambda)u_1. \end{aligned}$$

These equations are equivalent to

$$\begin{aligned} &(1 + \Theta G_R(\lambda))u_1 - \Theta G_I(\lambda)u_2 = 0, \\ &-\Theta G_I(\lambda)u_1 - (1 + \Theta G_R(\lambda))u_2 = 0. \end{aligned}$$

Since $(u_1, u_2) \neq (0, 0)$, the system has a nontrivial solution. It follows that the determinant of the system vanishes. A simple calculation shows that this determinant equals $(1 + \Theta G_R(\lambda))^2 + (\Theta G_I(\lambda))^2$. We conclude that Θ satisfies the above equations if and only if

$$\Theta G_I(\lambda) = 0 \quad \text{and} \quad \Theta G_R(\lambda) = -1.$$

It follows that $\Theta(G_R(\lambda) - \alpha G_I(\lambda)) = -1$ for all $\alpha \in \mathbf{R}$, so that we have $1 \leqslant \varepsilon\|G_R(\lambda) - \alpha G_I(\lambda)\|$. Hence we have

$$d(G_R(\lambda), \mathbf{R}G_I(\lambda)) \geqslant 1/\varepsilon.$$

Conversely, let us assume that $d(G_R(\lambda), \mathbf{R}G_I(\lambda)) \geqslant 1/\varepsilon$. By a duality theorem (see [6]) there exists a vector $z \in \mathbf{R}^t$, $\|z\| = 1$ such that

$$\begin{aligned} z^{\mathrm{T}}G_R(\lambda) &= d(G_R(\lambda), \mathbf{R}G_I(\lambda)), \\ z^{\mathrm{T}}G_I(\lambda) &= 0. \end{aligned}$$

Let us define $\Theta = -d(G_R(\lambda), \mathbf{R}G_I(\lambda))^{-1}z$ and $x = P(\lambda)^{-1}D$. In this case, we have $(P(\lambda) + D\Theta E(\lambda))$ $x = 0$. □

5. Conclusion

In this note, we have shown that the structured pseudospectrum is equal to the pseudospectrum for the following structures: Toeplitz, circulant, Hankel and symmetric. We have also shown that this result is false for the Hermitian and skew-Hermitian structures. We have generalized these results to pseudospectra of matrix polynomials with Toeplitz, circulant, Hankel and symmetric structures. Moreover, we have given a formula for structured pseudospectra of real matrix polynomials.

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