Amélioration de la précision et validation des algorithmes numériques : le cas du calcul des racines de polynômes

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A famous failure: Patriot missile

- 1/10 is not representable by a finite number of digits in basis 2
  $1/10 = 0.00011001100110011001100 11001100\ldots$

- On a 24 bit fixed-point register
  error $= 1.1001100\ldots \times 2^{-24} \approx 0.000000095$ in decimal

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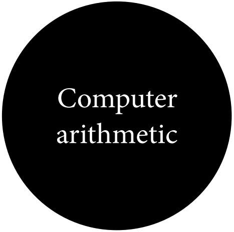
- First Gulf War in 1991: an american Patriot missile battery failed to intercept an Iraqi Scud missile. The Scud killed 28 soldiers.

- After 100 hours, the error is about 0.34 s: a Scud travels at about 1500 m/s, it makes 500 m

Research topic: compute *fast* and *accurately*



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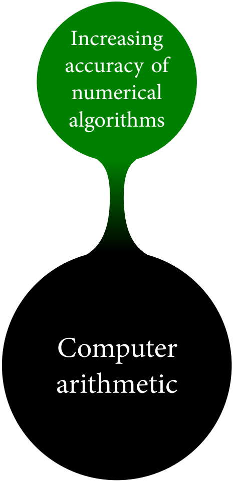

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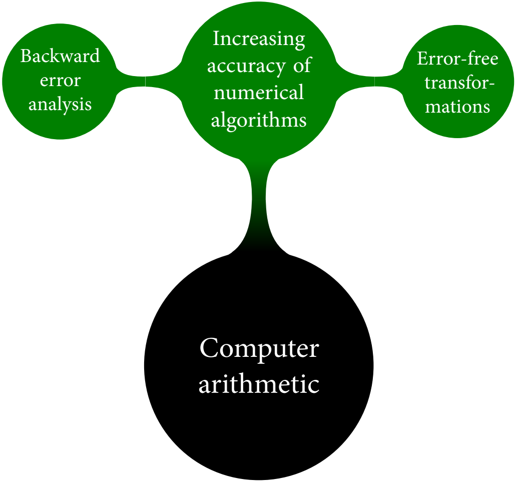

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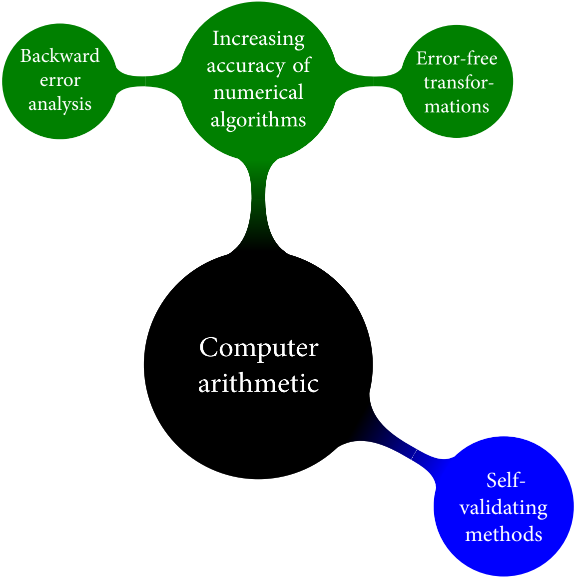

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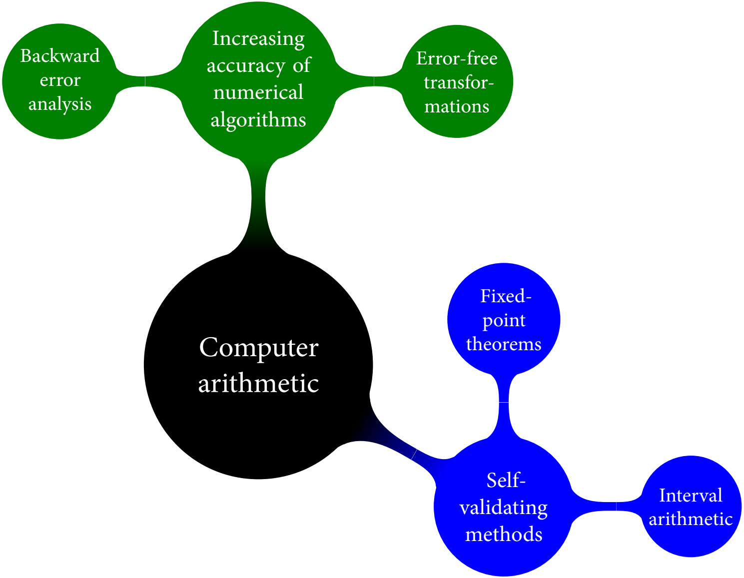

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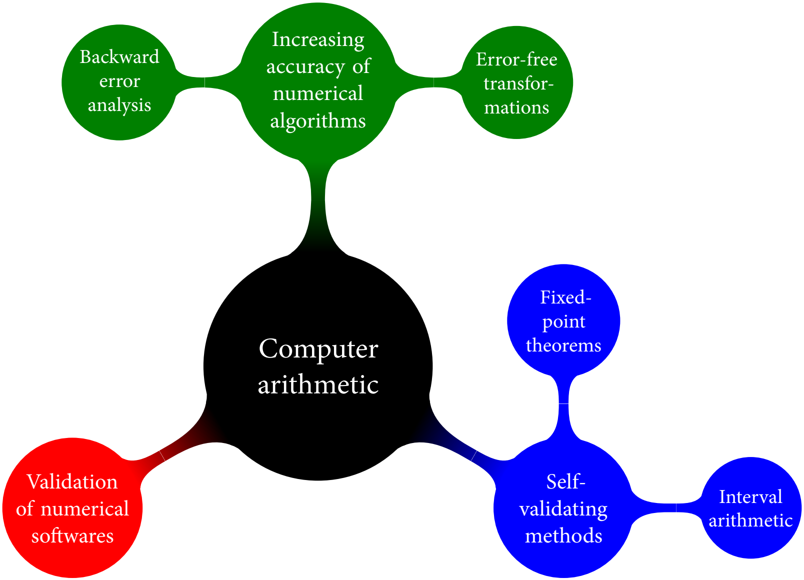

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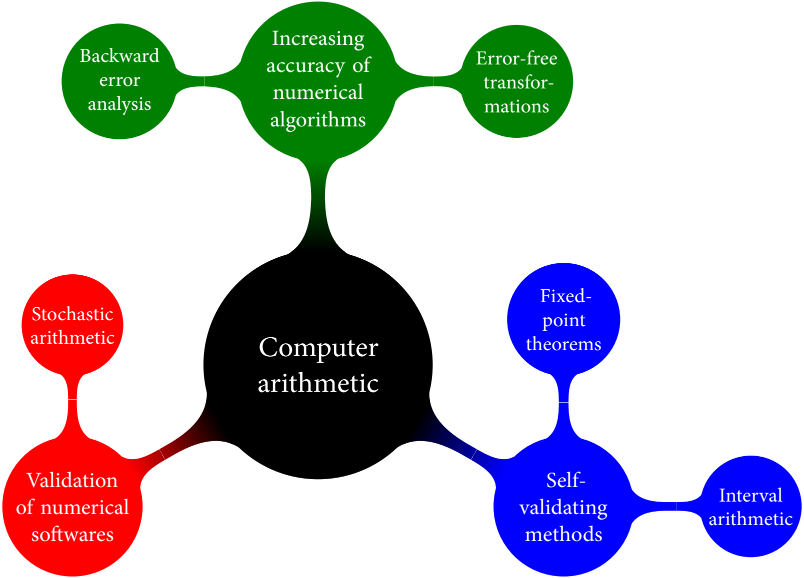

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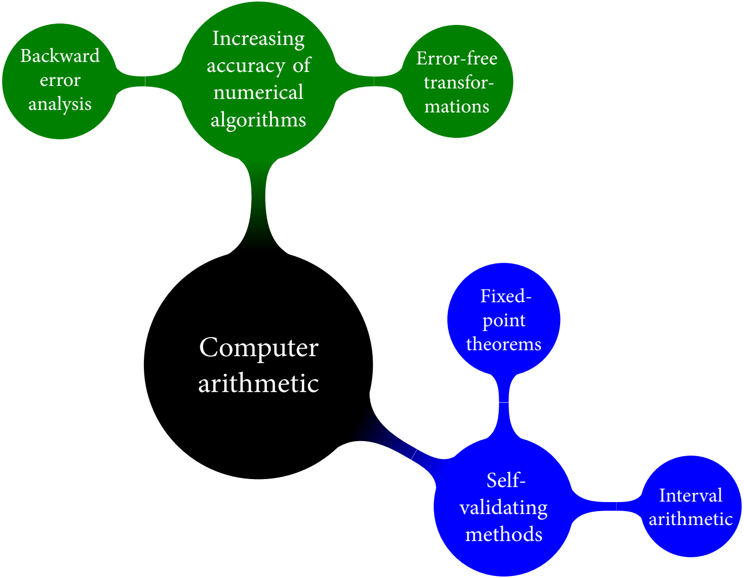

Outline

1. Introduction
2. Floating-point arithmetic
3. Error analysis and increase of accuracy
4. Interval analysis and self-validating methods
5. Conclusion and perspectives

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Standard model of floating-point arithmetic

Assume floating-point arithmetic adhering IEEE 754 with rounding to nearest with unit roundoff $\mathbf{u}$ (no underflow nor overflow)

Floating point system $\mathbb{F} \subset \mathbb{R}$:

$$x = \pm x_0.x_1 \dots x_{p-1} \times b^e, \quad 0 \le x_i \le b-1, \quad x_0 \ne 0$$

With $x, y \in \mathbb{F}$ and $\circ \in \{+, -, \times, /\}$, $x \circ y$ is not in general a floating-point number

$$\mathrm{fl}(x \circ y) = (x \circ y)(1+\delta), \quad |\delta| \le \mathbf{u}$$

IEEE 754 standard (1985,2008)

Type	Size	Significand	Exponent	Unit roundoff	Range
binary32	32 bits	23+1 bits	8 bits	$\mathbf{u} = 2^{-24} \approx 5.96 \times 10^{-8}$	$\approx 10^{\pm 38}$
binary64	64 bits	52+1 bits	11 bits	$\mathbf{u} = 2^{-53} \approx 1.11 \times 10^{-16}$	$\approx 10^{\pm 308}$

We denote: $\gamma_n := n\mathbf{u}/(1-n\mathbf{u}) \approx n\mathbf{u}$

Understanding the difficulties when computing with finite precision

- Controlling the effects of finite precision:
  - How to measure the difficulty of solving the problem?
  - How to characterize the reliability of the algorithm?
  - How to estimate the accuracy of the computed solution?
- Limiting the effects of finite precision
  - How to improve the accuracy of the solution?

How to answer these questions?

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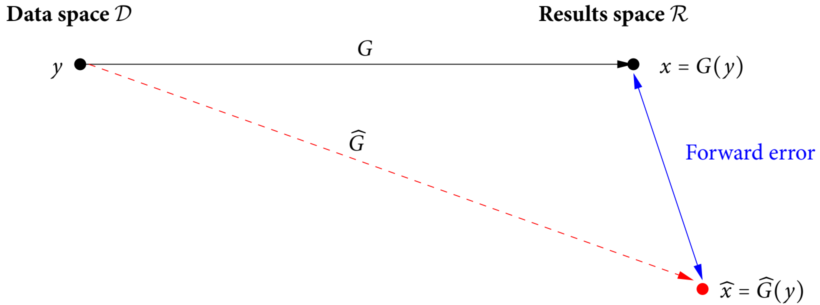
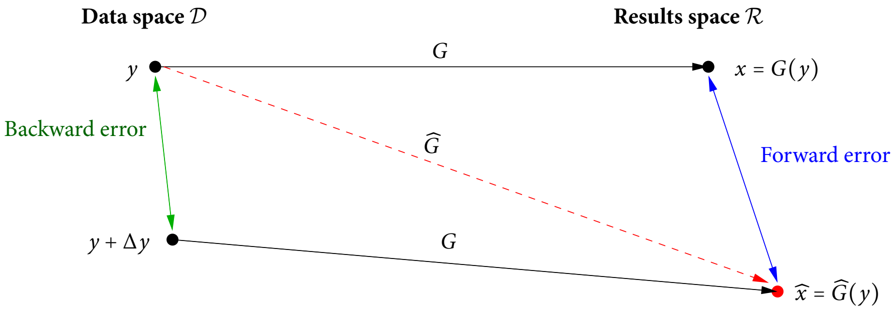
Error analysis (Wilkinson, Higham)



- Forward error analysis

Error analysis (Wilkinson, Higham)



- Forward error analysis
- Backward error analysis
  Identify $\widehat{x}$ as the solution of a perturbed problem: $\widehat{x} = G(y + \Delta y)$.

Advantages of backward error analysis

- **How to measure the difficulty of solving the problem ?**
  Condition number measures the sensitivity of the solution to perturbation in the data

  Condition number : $\mathrm{cond}(P, y) := \lim_{\varepsilon \to 0} \sup_{\|\Delta y\| \leq \varepsilon} \left\{ \frac{\|\Delta x\|_{\mathcal{R}}}{\|\Delta y\|_{\mathcal{D}}} \right\}$

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- **How to appreciate the reliability of the algorithm?**
  Backward error measures the distance between the problem we solved and the initial problem.

  Backward error : $\eta(\widehat{x}) = \min_{\Delta y \in \mathcal{D}} \{ \|\Delta y\|_{\mathcal{D}} : \widehat{x} = G(y + \Delta y) \}$

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  At first order, the rule of thumb:

  forward error $\lesssim$ condition number $\times$ backward error.

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Achieving more accuracy with compensated algorithms

Key tools for accurate computation

- fixed length expansions libraries: double-double (Briggs, Bailey, Hida, Li), quad-double (Bailey, Hida, Li)
- arbitrary length expansions libraries: Priest, Shewchuk
- arbitrary multiprecision libraries: MP, MPFUN/ARPREC, MPFR
- compensated algorithms (e.g. Kahan, Priest, Ogita-Rump-Oishi)

Error-free transformations (EFT) (Dekker, Knuth) are properties and algorithms to compute the elementary rounding errors,

$$a, b \in \mathbb{F}, \quad a \circ b = \mathrm{fl}(a \circ b) + e, \text{ and } e \in \mathbb{F}$$

EFT for addition

$$x = a \oplus b \quad \Rightarrow \quad a + b = x + y \quad \text{with } y \in \mathbb{F},$$

Algorithm of Dekker (1971) and Knuth (1974)

Algorithm (EFT of the sum of 2 floating-point numbers)

function $[x, y] = \texttt{TwoSum}(a, b)$

$x = a \oplus b$

$z = x \ominus a$

$y = (a \ominus (x \ominus z)) \oplus (b \ominus z)$

EFT for multiplication

$$x = a \otimes b \quad \Rightarrow \quad a \times b = x + y \quad \text{with } y \in \mathbb{F},$$

Given $a, b, c \in \mathbb{F}$,

- $\texttt{FMA}(a, b, c)$ is the nearest floating-point number $a \cdot b + c \in \mathbb{F}$

Algorithm (EFT of the product of 2 floating-point numbers)

function $[x, y] = \texttt{TwoProduct}(a, b)$
 $x = a \otimes b$
 $y = \texttt{FMA}(a, b, -x)$

The FMA is available for example on PowerPC, Itanium, Cell, Xeon Phi, Haswell processors.

Horner scheme

Algorithm

function $\texttt{res} = \texttt{Horner}(p, x)$ % $p(x) = \sum_{i=0}^{n} a_i x^i$

$s_n = a_n$

for $i = n-1 : -1 : 0$

$p_i = s_{i+1} \otimes x$

$s_i = p_i \oplus a_i$

end

$\texttt{res} = s_0$

Condition number for the evaluation of $p(x)$:

$$\mathrm{cond}(p, x) = \frac{\sum_{i=0}^{n} |a_i||x|^i}{|\sum_{i=0}^{n} a_i x^i|} = \frac{\widetilde{p}(|x|)}{|p(x)|}$$

Relative error bound: $$\frac{|p(x) - \texttt{Horner}(p, x)|}{|p(x)|} \leq \underbrace{\gamma_{2n}}_{\approx 2n\mathbf{u}} \mathrm{cond}(p, x)$$

Horner scheme

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$s_n = a_n$

for $i = n - 1 : -1 : 0$

$p_i = s_{i+1} \otimes x$ % rounding error π_i

$s_i = p_i \oplus a_i$ % rounding error σ_i

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EFT for Horner scheme

Algorithm (with Langlois, Louvet, JJIAM 2008)

function $[\texttt{Horner}(p,x), p_\pi, p_\sigma] = \texttt{EFTHorner}(p,x)$

$s_n = a_n$

for $i = n-1 : -1 : 0$

  $[p_i, \pi_i] = \texttt{TwoProduct}(s_{i+1}, x)$

  $[s_i, \sigma_i] = \texttt{TwoSum}(p_i, a_i)$

end

$\texttt{Horner}(p,x) = s_0$

$$p_\pi(x) = \sum_{i=0}^{n-1} \pi_i x^i, \qquad p_\sigma(x) = \sum_{i=0}^{n-1} \sigma_i x^i$$

$$\boxed{p(x) = \texttt{Horner}(p,x) + (p_\pi + p_\sigma)(x)}$$

Compensated Horner scheme (CHS) and its accuracy

Algorithm (CHS, with Langlois, Louvet, JJIAM 2008)

function `res` = `CompHorner`(p, x)
$[h, p_\pi, p_\sigma]$ = `EFTHorner`(p, x)
c = `Horner`$(p_\pi \oplus p_\sigma, x)$
`res` = $h \oplus c$

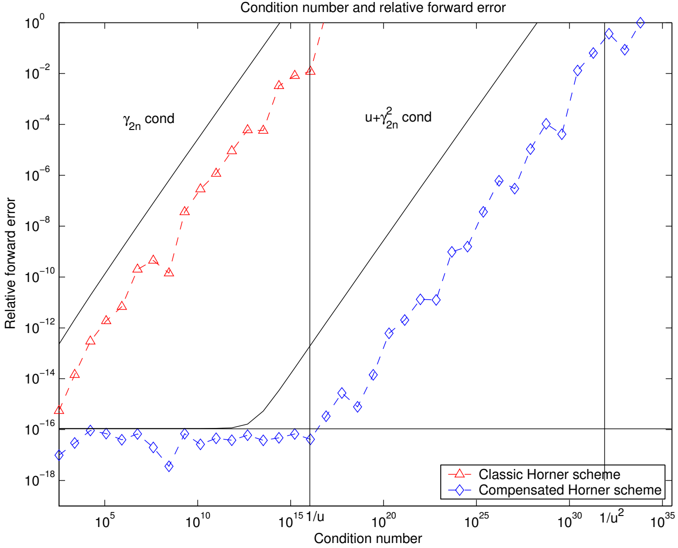
Theorem (with Langlois, Louvet, JJIAM 2008)

Let p be a polynomial of degree n with floating point coefficients, and x be a floating point value. Then if no underflow occurs,

$$\frac{|\mathtt{CompHorner}(p,x) - p(x)|}{|p(x)|} \leq \mathbf{u} + \underbrace{\gamma_{2n}^2}_{\approx 4n^2\mathbf{u}^2} \mathrm{cond}(p,x).$$

Numerical experiments: testing the accuracy

Evaluation of $p_n(x) = (x-1)^n$ for $x = \mathrm{fl}(1.333)$ and $n = 3, \ldots, 42$



Numerical experiments: testing the speed efficiency

We compare

- `Horner`: IEEE 754 double precision Horner scheme
- `CompHorner`: our Compensated Horner scheme
- `DDHorner`: Horner scheme with internal double-double computation

All computations are performed in C and IEEE 754 double precision

ratio	minimum	mean	maximum	theoretical
`CompHorner`/`Horner`	1.5	2.9	3.2	13
`DDHorner`/`Horner`	2.3	8.4	9.4	17

Compensated Horner Derivative algorithm

The Horner Derivative (HD) algorithm is the classic method for the evaluation of the k-derivative of a polynomial $p(x)$

Algorithm (HD)

function $\texttt{res=HD}(p, x, k)$
  $y_i^j = 0$ for $i = 0:1:k$ and $j = n+1:-1:0$
  $y_{-1}^{j+1} = a_j$ for $j = n:-1:0$

  for $j = n:-1:0$
    for $i = \min(k, n-j):-1:\max(0, k-j)$
      $y_i^j = x \otimes y_i^{j+1} \oplus y_{i-1}^{j+1}$
    end
  end
  $\texttt{res} = k! \otimes y_k^0$

Algorithm (CHD)

function $\texttt{res=CompHD}(p, x, k)$
  $y_i^j = 0$, $\widehat{\epsilon y}_i^j = 0$, for $i = 0:1:k$, and $j = n+1:-1:0$
  $y_{-1}^{j+1} = a_j$, $\widehat{\epsilon y}_{-1}^{j+1} = 0$, for $j = n:-1:0$
  for $j = n:-1:0$
    for $i = \min(k, n-j):-1:\max(0, k-j)$
      $[s, \pi_i^j] = \texttt{TwoProd}(x, \widehat{y}_i^{j+1})$
      $[\widehat{y}_i^j, \sigma_i^j] = \texttt{TwoSum}(s, \widehat{y}_{i-1}^{j+1})$
      $\widehat{\epsilon y}_i^j = x \otimes \widehat{\epsilon y}_i^{j+1} \oplus \widehat{\epsilon y}_{i-1}^{j+1} \oplus (\pi_i^j \oplus \sigma_i^j)$
    end
  end
  $\texttt{res} = (\widehat{y}_k^0 \oplus \widehat{\epsilon y}_k^0) \otimes k!$

(with Jiang and al., JCAM 2013)

Rounding error analysis of CHD algorithm

Theorem (with Jiang and al., JCAM 2013)

Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree n with floating-point coefficients, and x a floating-point value (with $p^{(k)}(x) \neq 0$). The relative forward error bound in CHD algorithm is such that

$$\frac{|\texttt{CompHD}(p,x,k) - p^{(k)}(x)|}{|p^{(k)}(x)|} \leq 2\mathbf{u} + (k+1)\underbrace{\gamma_{2n}\gamma_{3n}}_{\approx 6n^2\mathbf{u}^2}\texttt{cond}(p,x,k).$$

The condition number for the k-th derivative evaluation of a polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$ at entry x is given by

$$\texttt{cond}(p,x,k) = \frac{k!\sum_{m=k}^{n}\binom{m}{k}|x|^{m-k}|a_m|}{|k!\sum_{m=k}^{n}\binom{m}{k}x^{m-k}a_m|} = \frac{\widetilde{p}^{(k)}(x)}{|p^{(k)}(x)|},$$

Condition number for root finding

Definition (Chatelin et al., 1996)

Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n and x be a simple zero of p. The condition number of x is defined by

$$\mathtt{cond}_{\mathtt{root}}(p, x) = \lim_{\varepsilon \to 0} \sup \left\{ \frac{|\Delta x|}{\varepsilon |x|} : |\Delta a_i| \leq \varepsilon |a_i| \right\}.$$

Theorem (Chatelin et al., 1996)

Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n and x be a simple zero of p. The condition number of x is given by

$$\mathtt{cond}_{\mathtt{root}}(p, x) = \frac{\widetilde{p}(|x|)}{|x||p'(x)|},$$

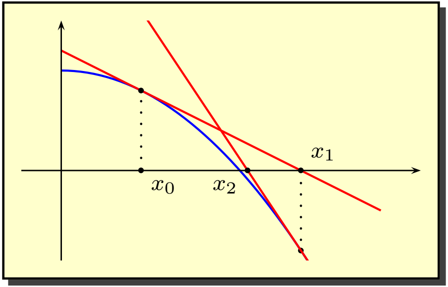
with $\widetilde{p}(x) = \sum_{i=0}^{n} |a_i| z^i$.

Application to Newton's method

Algorithm (The classic Newton's method)

$$x_0 = \xi$$
$$x_{i+1} = x_i - \frac{\mathtt{Horner}(p,x_i)}{\mathtt{HD}(p,x_i,1)}$$

$$\frac{|x_{i+1} - x|}{|x|} \approx \gamma_{2n}\,\mathrm{cond}_{\mathtt{root}}(p, x) \qquad \text{[Higham, 1996]}$$



Application to Newton's method

Algorithm (The accurate Newton's method)

$$x_0 = \xi$$
$$x_{i+1} = x_i - \frac{\texttt{CompHorner}(p, x_i)}{\texttt{HD}(p, x_i, 1)}$$

Theorem (Graillat, CMWA 2008)

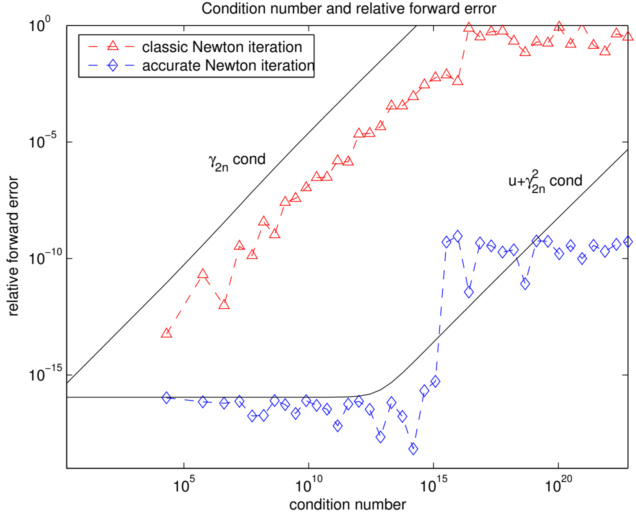
Assume that there is an x such that $p(x) = 0$ and $p'(x) \neq 0$ is not too small. Assume also that $\mathbf{u} \cdot \texttt{cond}_{\texttt{root}}(p, x) \leq 1/8$.
Then, for all x_0 such that $\beta |p'(x)^{-1}| |x_0 - x| \leq 1/8$, Newton's method in floating point arithmetic generates a sequence of $\{x_i\}$ whose relative error decreases until the first i for which

$$\frac{|x_{i+1} - x|}{|x|} \approx \mathbf{u} + \gamma_{2n}^2 \, \mathrm{cond}_{\texttt{root}}(p, x).$$

Application to Newton's method

Test with $p_n(x) = (x-1)^n - 10^{-8}$ and $x = 1 + 10^{-8/n}$ for $n = 1:40$

$\mathrm{cond}(p_n, x)$ varies from 10^4 to 10^{22}



Application to Newton's method

Algorithm (The new accurate Newton's method)

$$x_0 = \xi$$
$$x_{i+1} = x_i - \frac{\texttt{CompHorner}(p,x_i)}{\texttt{CompHD}(p,x_i,1)}$$

We proved that

- that the convergence of iterations strongly depends on the accuracy of the derivative's evaluation when the problem of finding simple root is too ill-conditioned, and
- that the accuracy of the final iteration result depends on the accuracy with which the residual is computed.

Application to Newton's method

We have shown (with Jiang and al., JCAM 2013) that

In case of classic Newton's algorithm:

$$\left|\frac{x_i - x}{x}\right| < C\gamma_{2n}\mathtt{cond}_{\mathtt{root}}(p, x).$$

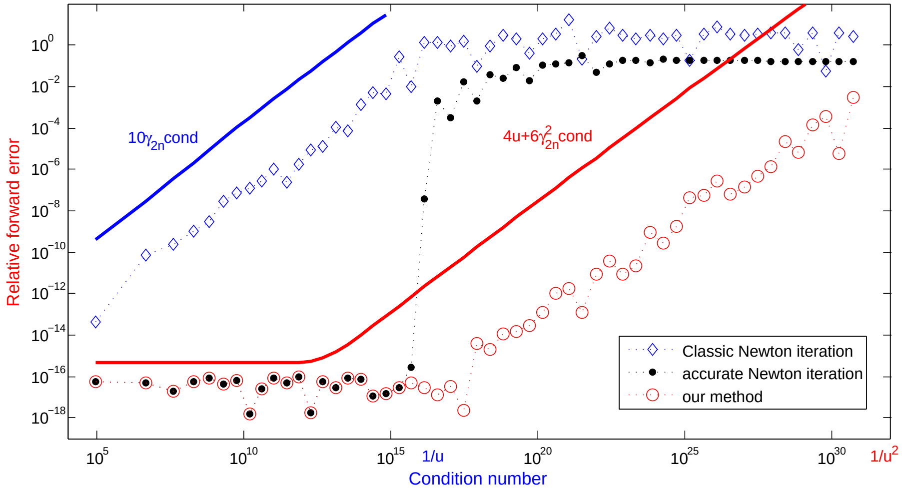
In case of accurate Newton's algorithms:

$$\left|\frac{x_i - x}{x}\right| < Ku + D\gamma_{2n}^2\mathtt{cond}_{\mathtt{root}}(p, x).$$

where C, K and D are small factors.

Numerical experiments

Simple real zero of the expanded form of the polynomial
$p_n(x) = (x-1)^n - 2^{-31}$, for $n = 2:55$



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Interval arithmetic

- Interval arithmetic: replace numbers by intervals and compute.
- Fundamental theorem of interval arithmetic: the exact result belongs to the computed interval.
- No result is lost, the computed interval is guaranteed to contain every possible result.

Definitions and notation

Objects

- interval of real numbers: closed connected sets of $\mathbb{R}$
  - interval for π: $[3.14159, 3.14160]$
  - data d known with absolute uncertainty of ε: $[d - \varepsilon, d + \varepsilon]$
- interval vector

$$\boldsymbol{v} = \begin{pmatrix} [1,2] \\ [2,4] \end{pmatrix}$$

- interval matrix

$$\boldsymbol{A} = \begin{pmatrix} [1,3] & [3,4] \\ [2,5] & [1,2] \end{pmatrix}$$

Representation inf-sup of intervals

$$\boldsymbol{x} = [\underline{x}; \overline{x}] = \{x \in \mathbb{R} : \underline{x} \le x \le \overline{x}\}.$$

The set of interval of $\mathbb{R}$ is denoted $\mathbb{IR}$.

Operations on intervals

- Given two intervals $\boldsymbol{x}$, $\boldsymbol{y}$ and $\diamond \in \{+, -, \times, /\}$, one defines

$$\boldsymbol{x} \diamond \boldsymbol{y} = \{x \diamond y : x \in \boldsymbol{x}, y \in \boldsymbol{y}\}.$$

- On can show for example that :

$$\begin{aligned}
\boldsymbol{x} + \boldsymbol{y} &= [\underline{x} + \underline{y}; \overline{x} + \overline{y}], \\
\boldsymbol{x} \times \boldsymbol{y} &= [\min(\underline{x}\,\underline{y}, \underline{x}\,\overline{y}, \overline{x}\,\underline{y}, \overline{x}\,\overline{y}), \max(\underline{x}\,\underline{y}, \underline{x}\,\overline{y}, \overline{x}\,\underline{y}, \overline{x}\,\overline{y})],
\end{aligned}$$

- In floating-point arithmetic, if one wants validated results, one needs to take into account rounding errors:

$$\boldsymbol{x} + \boldsymbol{y} = [\nabla(\underline{x} + \underline{y}), \Delta(\overline{x} + \overline{y})] \supseteq \{x + y | x \in \boldsymbol{x}, y \in \boldsymbol{y}\}$$

where ∇ (resp. Δ) denotes rounding toward $-\infty$ (resp. $+\infty$).

Proving that a matrix is nonsingular

Theorem (classic)

Let $A \in \mathbb{R}^{n \times n}$ be a matrix and $R \in \mathbb{R}^{n \times n}$ another matrix such that $\|I - RA\| < 1$. Then A is nonsingular

On a computer, choose for $R \approx A^{-1}$ and then compute $\|I - RA\|$ with interval arithmetic.

```
R = inv(A)
C = eye(n) - R*intval(A)
nonsingular = ( norm(C,1) < 1 )
```

If nonsingular = 1, then A is nonsingular.
If nonsingular = 0, then we can say nothing

A simple approach

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\widehat{x} \in \mathbb{R}^n$ unknown such that $f(\widehat{x}) = 0$

Let $\widetilde{x} \approx \widehat{x}$ such that $f(\widetilde{x}) \approx 0$

Find a bound for $\widetilde{x}$: an interval X such that $\widehat{x} \in X$

We have

$$f(x) = 0 \qquad \Leftrightarrow \qquad g(x) = x$$

with $g(x) := x - Rf(x)$ with $\det(R) \neq 0$.

Theorem (Brouwer, 1912)

Every continuous function from a closed ball of a Euclidean space to itself has a fixed point.

A simple approach (cont'd)

By Brouwer fixed point theorem,

$$X \in \mathbb{IR}^n, \quad g(X) \subseteq X \quad \Rightarrow \quad \exists \widehat{x} \in X,\ g(\widehat{x}) = \widehat{x} \quad \Rightarrow \quad f(\widehat{x}) = 0$$

We just have to check $g(X) \subseteq X$ and prove $\det(R) \neq 0$.

But naive approach fails:

$$g(X) \subseteq X - Rf(X) \not\subseteq X$$

Bounds for the solution of nonlinear systems

Theorem (Rump, 1983)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f = (f_1, \ldots, f_n) \in \mathcal{C}^1$, $\tilde{x} \in \mathbb{R}^n$, $X \in \mathbb{IR}^n$ with $0 \in X$ and $R \in \mathbb{R}^{n\times n}$ be given. Let $M \in \mathbb{IR}^{n\times n}$ be given such that

$$\{\nabla f_i(\zeta) : \zeta \in \tilde{x} + X\} \subseteq M_{i,:}\ .$$

Assume

$$-Rf(\tilde{x}) + (I - RM)X \subseteq int(X).$$

Then there is a unique $\hat{x} \in \tilde{x} + X$ with $f(\hat{x}) = 0$. Moreover, every matrix $\widetilde{M} \in M$ is nonsingular. In particular, the Jacobian $J_f(\hat{x}) = \frac{\partial f}{\partial x}(\hat{x})$ is nonsingular.

Verification of multiple roots

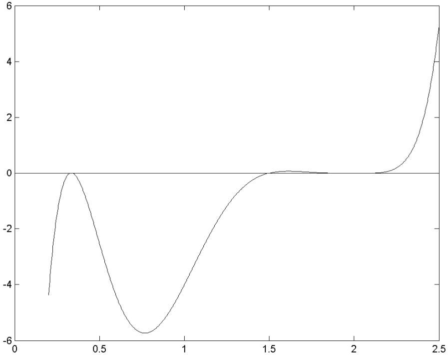
- Verification method for computing guaranteed (real or complex) error bounds for double roots of systems of nonlinear equations.
- To circumvent the problem of ill-posedness we prove that a slightly perturbed system of nonlinear equations has a double root.
- For example, for a given univariate function $f : \mathbb{R} \to \mathbb{R}$ we compute two intervals $X, E \subseteq \mathbb{R}$ with the property that there exists $\widehat{x} \in X$ and $\widehat{e} \in E$ such that $\widehat{x}$ is a double root of $\overline{f}(x) := f(x) - \widehat{e}$.

Verification of multiple roots

The typical scenario in the univariate case is a function $f : \mathbb{R} \to \mathbb{R}$ with a double root $\widehat{x}$, i.e. $f(\widehat{x}) = f'(\widehat{x}) = 0$ and $f''(\widehat{x}) \neq 0$.

Consider, for example,

$$\begin{aligned} f(x) &= 18x^7 - 183x^6 + 764x^5 - 1675x^4 + 2040x^3 - 1336x^2 + 416x - 48 \\ &= (3x-1)^2(2x-3)(x-2)^4 \end{aligned}$$



Verification of multiple roots

- Verification methods for multiple roots of polynomials already exist (Rump, 2003). A set containing k roots of a polynomial is computed, but no information on the true multiplicity can be given.

- Algorithm `verifypoly` in INTLAB. Computing inclusions `X1`, `X2` and `X3` of the simple root $x_1 = 1.5$, the double root $x_2 = 1/3$ and the quadruple root $x_3 = 2$ of f:

```
>> X1 = verifypoly(f,1.3), X2 = verifypoly(f,.3), X3 = verifypoly(f,2.1)
intval X1 =
[   1.49999999999904,   1.50000000000078]
intval X2 =
[   0.33333316656015,   0.33333343640539]
intval X3 =
[    1.99741678159164,   2.00363593397305]
```

Verification of multiple roots (cont'd)

- The accuracy of the inclusion of the double root $x_2 = 1/3$ is much less than that of the simple root $x_1 = 1.5$, and this is typical.

- Perturb f into $\widetilde{f}(x) := f(x) - \varepsilon$ for some small real constant ε and look at a perturbed root $\widetilde{f}(\widehat{x} + h)$ of $\widetilde{f}$, then

$$0 = \widetilde{f}(\widehat{x} + h) = -\varepsilon + \frac{1}{2} f''(\widehat{x}) h^2 + \mathcal{O}(h^3)$$

  implies

$$h \sim \sqrt{2\varepsilon / f''(\widehat{x})}.$$

- A relative error of size $\varepsilon \approx 10^{-16}$ implies a relative accuracy of $\sqrt{\varepsilon} \approx 10^{-8}$.

Dealing with double roots

- We consider for a double root the nonlinear system $G : \mathbb{R}^2 \to \mathbb{R}$ with

$$G(x, e) = \begin{pmatrix} f(x) - e \\ f'(x) \end{pmatrix} = 0$$

in the two unknowns x and e.

- The Jacobian of this system is

$$J_G(x, e) = \begin{pmatrix} f'(x) & -1 \\ f''(x) & 0 \end{pmatrix},$$

so that the nonlinear system is well-conditioned for the double root $x_2 = 1/3$ of f.

Dealing with double roots (cont'd)

- We provide (with Rump, NA 2010) an algorithm `verifynlss` in INTLAB.

```
>> Y2 = verifynlss(G,[.3;0])
intval Y2 =
[  3.333333333333328e-001,  3.333333333333337e-001]
[ -2.131628207280424e-014,  2.131628207280420e-014]
```

- This proves that there is a constant ε with $|\varepsilon| \leq 2.14 \cdot 10^{-14}$ such that the nonlinear equation $f(x) - \varepsilon = 0$ has a double root $\widehat{x}$ with $0.3333333333333328 \leq \widehat{x} \leq 0.3333333333333337$.

Outline

1 Introduction

2 Floating-point arithmetic

3 Error analysis and increase of accuracy

4 Interval analysis and self-validating methods

5 Conclusion and perspectives

Conclusion

- Compensated methods are a fast way to get accurate results
  They can be used to accurately evaluate residuals

- Self-validating methods can be an efficient and fast alternatives to obtain exact/certified results with finite precision arithmetic

Perspectives

- Increasing the accuracy:
  - a faithfully/correctly rounded Horner scheme
  - a correctly rounded 2-norm for vectors
  - "optimal" error bounds: like $|\mathrm{fl}(x^n) - x^n| \le (n-1)\mathbf{u}x^n$
- Reproducibility and accuracy for exascale computing:
  - implementation of Kulisch accumulator on GPU and Xeon Phi
  - efficient exact dot product for multiprecision interval arithmetic
  - efficient reproducible BLAS librairies on GPU and Xeon Phi
- Numerical validation, certification and symbolic-numeric computation:
  - self-validating methods to detect singularities (approximate gcd, singular matrix, etc.)
  - automatic tool using stochastic arithmetic and delta-debugging for results with given accuracy