Computation of stability radius for polynomials

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Abstract

A polynomial is stable if all its roots have negative real part, and unstable otherwise. For a stable polynomial, the distance to the nearest unstable polynomial is an important parameter in control theory for example. In this paper, we focus on this distance called the stability radius of polynomial p. We propose to modify the level contour function of the pseudozero set to derive a bisection algorithm that computes an arbitrary accurate approximation of this stability radius. Numerical simulations and comparisons with pseudozero graphics are here after presented.

Key words: pseudozero, abscissa mapping, stability radius, robust stability, bisection

AMS Subject Classifications: 93D09, 30C10, 30C15, 12D10, 26C10

1 Introduction

In control theory, a classical transfer function is written as $H(p) = N(p)/D(p)$ where N and D are polynomials and p a parameter of the system. The system described with the function H is stable if all the zeros of D have negative real part, that is if the polynomial D is stable. Since uncertainty in the coefficients

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of the polynomial are unavoidable in most real problem (data uncertainty, rounding error), it is useful for a stable system to measure the distance to the nearest unstable system *i.e.* the distance to the nearest unstable polynomial from D.

Using companion matrix, this polynomial problem could be reformulated as a matrix problem. A matrix $A \in \mathbf{C}^{n\times n}$ is stable if all its eigenvalues have negative real part, and unstable otherwise. When A is stable, the computation of the distance (with respect to a matrix norm $\|\cdot\|$) from A to the nearest unstable matrix,

$$\beta(A) = \min\{\|E\| : A+E \in \mathbf{C}^{n\times n} \text{ is unstable}\}.$$

has been intensively studied in numerical linear algebra [2, 4, 5].

To ensure that this minimum distance is the associated polynomial radius, the perturbed matrix $A+E$ has to conserve the companion structure of A. Up to our knowledge, no existing matrix algorithm guarantees this property.

In this paper, we proposed to compute the minimum distance to an unstable polynomial staying in the field of polynomials. The key tool to succeed is the polynomial pseudozero set introduced by Mosier (see [8]).

Let $\mathcal{P}_n$ be the linear space of polynomials of degree n or less with complex coefficients and $\mathcal{M}_n$ the subset of monic polynomials of degree n. Let $p \in \mathcal{M}_n$ given by

$$p(z) = \sum_{i=0}^{n} p_i z^i, \quad p_n = 1. \tag{1}$$

Representing p by the vector $(p_0, \ldots, p_{n-1})$ of its coefficients, we identify the norm $\|\cdot\|$ on $\mathcal{M}_n$ to the 2-norm on $\mathbf{C}^n$ of the corresponding vector.

An ε-neighborhood of p is the set of all polynomials of $\mathcal{M}_n$, closed enough to p, that is,

$$N_\varepsilon(p) = \{\widehat{p} \in \mathcal{M}_n : \|p - \widehat{p}\| \le \varepsilon\}. \tag{2}$$

Then the ε-pseudozero set of p is defined to include all the zeros of the ε-neighborhood of p. A non constructive definition of this set is

$$Z_\varepsilon(p) = \{z \in \mathbf{C} : \widehat{p}(z) = 0 \text{ for } \widehat{p} \in N_\varepsilon(p)\}. \tag{3}$$

In the two first sections, we recall the notion of pseudozero set and we give some definitions about polynomials. In the third section, we propose a bisection algorithm to compute the stability radius. In the last section, we present some numerical simulations.

2 Pseudozero set and stability radius

Following Proposition 1 provides a computable counterpart of this definition.

Proposition 1. *The ε-pseudozero set of p verifies*

$$Z_\varepsilon(p) = \left\{ z \in \mathbf{C} : g(z) := \frac{|p(z)|}{\|\underline{z}\|} \leq \varepsilon \right\}, \tag{4}$$

where $\underline{z} = (1, z, \ldots, z^{n-1})$.

This proposition was proved in [11] for the 2-norm and in [3, 10] for a general norm. For completeness of this paper, we recall the proof.

Proof. If $z \in Z_\varepsilon(p)$ then there exists $\widehat{p} \in \mathcal{M}_n$ such that $\widehat{p}(z) = 0$ et $\|p - \widehat{p}\| \leq \varepsilon$. From Hölder's inequality $|x^t y| \leq \|x\|\|y\|$, we get

$$|p(z)| = |p(z) - \widehat{p}(z)| = |\sum_{i=0}^{n} (p_i - \widehat{p}_i) z^i| \leq \|p - \widehat{p}\|\|\underline{z}\|.$$

It follows that $|p(z)| \leq \varepsilon \|\underline{z}\|$.

To prove the reciprocal, let $u \in \mathbf{C}$ be such that $|p(u)| \leq \varepsilon \|\underline{u}\|$. If $u \neq 0$, we can write $u = |u| e^{i\theta}$, $\theta \in [0, 2\pi[$. Let us introduce the polynomials r and p_u defined by

$$\begin{aligned} r(z) &= \sum_{k=0}^{n-1} r_k z^k \text{ with } r_k = |u|^k e^{-ik\theta}, \\ p_u(z) &= p(z) - \frac{p(u)}{r(u)} r(z). \end{aligned}$$

The polynomial p_u is the nearest polynomial of p, in the sense of the norm $\|\cdot\|$, with u as root.
It is clear that $r(u) = \|\underline{u}\|^2 = \|r\|^2$, and $p_u(u) = 0$. So we have

$$\|p - p_u\| = \frac{|p(u)|}{|r(u)|} \|r\| \leq \varepsilon.$$

Thus, we obtain

$$\|p - p_u\| \leq \varepsilon.$$

Hence $u \in Z_\varepsilon(p)$.

If now $u = 0$. Let us define $p_u(z) = p(z) - p(u)$. It is clear that $p_u(u) = 0$. Besides, $\|p - p_u\| = |p(u)| \leq \varepsilon$ by hypothesis. In the same way, $u \in Z_\varepsilon(p)$. ☐

Considered polynomials have been chosen to be monic polynomials to ensure the ε-pseudozero set is bounded.

Proposition 2. *The ε-pseudozero set is a compact set contained in the ball of center O and radius $1 + \|p\| + \varepsilon$.*

Proof. As the function g is continuous, the set $Z_\varepsilon(p) = g^{-1}([0, \varepsilon])$ is closed. Let us denote by $\{z_j\}_{j=1:n}$ the roots of the polynomial p counted with their multiplicities and $r = \max_j |z_j|$. It is well known (see [7, p.154]) that

$$r \leq 1 + \|p\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the ∞-norm of p considered as the vector $(p_0, \ldots, p_{n-1})$ of $\mathbf{C}^n$. If $z \in Z_\varepsilon(p)$ then there exists $\widehat{p} \in \mathcal{M}_n$ satisfying both $\widehat{p}(z) = 0$ and $\|p - \widehat{p}\| \leq \varepsilon$. It follows that $|z| \leq 1 + \|\widehat{p}\|_\infty$. Besides, we have $|\|\widehat{p}\|_\infty - \|p\|_\infty| \leq \|\widehat{p} - p\| \leq \varepsilon$ and so $\|\widehat{p}\|_\infty \leq \|p\|_\infty + \varepsilon$. Hence we obtain that $|z| \leq 1 + \|p\|_\infty + \varepsilon$. As $\|\cdot\|_\infty \leq \|\cdot\|$, we bound the radius of the ε-pseudozero set as follows $|z| \leq 1 + \|p\| + \varepsilon$. □

We introduce the function $h_{p,\varepsilon} : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$h_{p,\varepsilon}(x, y) = |p(x + iy)|^2 - \varepsilon^2 \sum_{j=0}^{n-1} (x^2 + y^2)^j. \tag{5}$$

It is clear that for a fixed x_0, the function $h_{p,\varepsilon}(x_0, y)$ is a polynomial of degree $2n$. In the same way, for a fixed y_0, the function $h_{p,\varepsilon}\varepsilon(x, y_0)$ is also a polynomial of degree $2n$. From Proposition 1, the pseudozero set $Z_\varepsilon(p)$ verifies

$$Z_\varepsilon(p) = \{(x, y) \in \mathbf{R}^2 : h_{p,\varepsilon}(x, y) \leq 0\}.$$

and we have the following proposition.

Proposition 3. *We have $h_{p,\varepsilon}(x, y) = 0$ if and only if there exists $q \in \mathcal{M}_n$ such that $q(x + iy) = 0$ and $\|p - q\| = \varepsilon$.*

Proof. It follows immediately from Proposition 1. □

3 Abscissa mapping and stability radius

We present three important notions and some relations between them.

The first one is the *abscissa mapping* for a polynomial $p \in \mathcal{M}_n$ defined by

$$a(p) = \max\{\mathrm{Re}(z) : p(z) = 0\}.$$

Hence, a stable polynomial satisfies $a(p) < 0$. The abscissa mapping

$$a : \mathcal{P}_n \to \mathbf{R}$$

defined by $a(p) = \max\{\mathrm{Re}(z) : p(z) = 0\}$ is continuous on $\mathcal{M}_n$. It is clear that a is not continuous on $\mathcal{P}_n$ as it is shown in [1]. Indeed, let us consider the polynomial $q_t(z) = (1 - tz)p(z)$ where p is a polynomial of degree less than n. We have $q_t \to p$ when $t \to 0$, whereas $a(q_t) = 1/t$.

To prove the continuity of a on $\mathcal{M}_n$, we will use the following result known as "the continuous dependence of the zeroes of a polynomial on its coefficients". The proof can be found in [6, 9].

Proposition 4. *Let*

$$p(z) = p_0 + p_1 z + \cdots + p_{n-1} z^{n-1} + z^n$$

be a monic polynomial with complex coefficients. Then, for every $\varepsilon > 0$*, there is* $\eta > 0$ *such that for any polynomial*

$$q(z) = q_0 + q_1 z + \cdots + q_{n-1} z^{n-1} + z^n$$

satisfying

$$\max_{0 \le i \le n} |p_i - q_i| < \eta,$$

we have

$$\min_{\sigma \in \mathfrak{S}_n} \max_{1 \le j \le n} |x_j - y_{\sigma(j)}| < \varepsilon,$$

where (x_j) *and* (y_j)*,* $j = 1, \ldots, n$*, are respectively the zeroes of* p *and* q*.*

We can now prove the continuity of a on $\mathcal{M}_n$.

Proposition 5. *The abscissa mapping*

$$a : \mathcal{P}_n \to \mathbf{R}$$

defined by $a(p) = \max\{\mathrm{Re}(z) : p(z) = 0\}$ *is continuous on* $\mathcal{M}_n$*.*

Proof. Let p in $\mathcal{M}_n$ and $\varepsilon > 0$. From Proposition 4, there is $\eta > 0$ such that for any $q \in \mathcal{M}_n$ satisfying

$$\max_{0 \le i \le n} |p_i - q_i| < \eta,$$

we have

$$\min_{\sigma \in \mathfrak{S}_n} \max_{1 \le j \le n} |x_j - y_{\sigma(j)}| < \varepsilon,$$

where (x_j) and (y_j), $j = 1, \ldots, n$, are respectively the zeros of p and q. It means that there is a permutation σ such that

$$\max_{1 \le j \le n} |\operatorname{Re}(x_j) - \operatorname{Re}(y_{\sigma(j)})| \le \max_{1 \le j \le n} |x_j - y_{\sigma(j)}| < \varepsilon.$$

We have

$$\begin{aligned} |a(q) - a(p)| &= |\max_{1 \le j \le n} \operatorname{Re}(y_j) - \max_{0 \le j \le n} \operatorname{Re}(x_j)| \\ &= |\max_{1 \le j \le n} \operatorname{Re}(y_{\sigma(j)}) - \max_{1 \le j \le n} \operatorname{Re}(x_j)| \\ &\le \max_{1 \le j \le n} |\operatorname{Re}(y_{\sigma(j)}) - \operatorname{Re}(x_j)| \\ &\le \varepsilon. \end{aligned}$$

We have proved the continuity of a on $\mathcal{M}_n$. $\square$

A natural extension of the abscissa mapping when polynomials are perturbed is the *pseudozeros abscissa mapping* defined by

$$a_\varepsilon(p) = \max\{\operatorname{Re}(z) : z \in Z_\varepsilon(p)\}.$$

Instead of computing the real part of the zeros of p, we are interesting in the real part of the ε-pseudozeros of p.

The third parameter is the distance of a given polynomial from the set of unstable polynomials. Such a *stability radius* is defined by

$$\beta(p) = \min\{\|p - q\| : q \in \mathcal{M}_n \text{ and } a(q) \ge 0\}.$$

Since the set of polynomials which are unstable is closed (due to the continuity of a on $\mathcal{M}_n$), the minimum is attained. We can know reformulate the stability radius $\beta(p)$ in term of pseudozeros. The stability radius is the largest ε for which the pseudozero set $Z_\varepsilon(p)$ lies in the left half-plane.

These notions are linked with the following relation

$$a_\varepsilon(p) \ge 0 \Longleftrightarrow \beta(p) \le \varepsilon.$$

Indeed if $a_\varepsilon(p) \ge 0$ then there exists $q \in \mathcal{M}_n$ such that $\|p - q\| \le \varepsilon$ and $z \in \mathbf{C}$ that satisfy $\operatorname{Re}(z) \ge 0$ and $q(z) = 0$. By definition of β, we can write that $\beta(p) \le \varepsilon$. Conversely, if $\beta(p) \le \varepsilon$ then there exists $q \in \mathcal{M}_n$ such that $a(q) \ge 0$, $\|p - q\| \le \varepsilon$ and q has at least a root $z \in \mathbf{C}$ with $\operatorname{Re}(z) \ge 0$. It follows that $a_\varepsilon(p) \ge 0$.

We may now state the following result.

Proposition 6. *If $p \in \mathcal{M}_n$ is not stable, we have $\beta(p) = 0$. Otherwise, when p is stable, we have $a_{\beta(p)}(p) = 0$.*

Proof. The first assertion is clear. For the second one, using the above result for $\varepsilon = \beta(p)$, we obtain $a_{\beta(p)}(p) \geq 0$. If we suppose that $a_{\beta(p)}(p) > 0$, then by continuity of $a_\varepsilon(p)$ with respect to ε, there exists $\varepsilon \in]0, \beta(p)[$ such that $a_\varepsilon(p) \geq 0$ which is equivalent to $\beta(p) \leq \varepsilon < \beta(p)$. This is a contradiction. Let us proof the continuity of $a_\varepsilon(p)$ with respect to ε. We have

$$\begin{aligned} a_\varepsilon(p) &= \sup\{a(q) : \|q - p\| \leq \varepsilon\} = \sup\{a(p + q) : \|q\| \leq \varepsilon\} \\ &= \sup\{a(p + \varepsilon r) : r \in \mathcal{M}_n \text{ and } \|r\| \leq 1\} = a(p + \varepsilon r_0), \end{aligned}$$

where $r_0 \in \mathcal{M}_n$ by continuity of a on $\mathcal{M}_n$ and the compactness of $\{r \in \mathcal{M}_n : \|r\| \leq 1\}$. □

In the sequel, we consider that polynomial p is stable. We have the following proposition.

Proposition 7. *The stability radius $\beta(p)$ satisfies*

$$\beta(p) = \min\{\|p - q\| : q \in \mathcal{M}_n \text{ and } a(q) = 0\}.$$

Proof. We prove that the polynomial q^* which realize the minimum value of $\beta(p)$ (*i.e.* $\|p - q^*\| = \beta(p)$) satisfies $a(q^*) = 0$. Indeed, if $a(q^*) > 0$, then let $p_t(z) = (1 - t)p(z) + tq(z)$, $t \in [0, 1]$. We clearly have $p_t \in \mathcal{M}_n$, $p_0(z) = p(z)$ and $p_1(z) = q^*(z)$. Let $\varphi : [0, 1] \to \mathbf{R}$ be the function $t \mapsto a(p_t)$. By continuity of a on $\mathcal{M}_n$ (see Proposition 5), the function φ is continuous. As $a(p) = \varphi(0) < 0$ and $a(q^*) = \varphi(1) \geq 0$, there is $\bar{t} \in]0, 1[$ such that $a(p_{\bar{t}}) = 0$. Besides, $\|p_{\bar{t}} - p\| = \bar{t}\|p - q^*\| < \|p - q^*\| = \beta(p)$. This is a contradiction. To conclude, we have $\beta(p) = \min\{\|p - q\| : q \in \mathcal{M}_n \text{ and } a(q) = 0\}$. □

The proposed algorithm that computes the stability radius of a given polynomial relies on the following theorem.

Theorem 1. *The equation $h_{p,\varepsilon}(0, y) = 0, y \in \mathbf{R}$, has a solution if and only if $\beta(p) \leq \varepsilon$.*

Proof. If the equation $h_{p,\varepsilon}(0, y) = 0, y \in \mathbf{R}$, has a solution u, it means by Proposition 3 that there exists a polynomial $\widehat{p}$ such that $\widehat{p}(iu) = 0$ and $\|p - \widehat{p}\| = \varepsilon$. By definition of $\beta(p)$, it implies that $\beta(p) \leq \varepsilon$.

If now $\beta(p) \leq \varepsilon$, there exists a polynomial q such that $\|q - p\| \leq \varepsilon$ and $a(q) \geq 0$. Therefore, at least one root of q has a positive real part. Let us define the polynomial $p_t(z) = (1-t)p(z) + tq(z)$, $t \in [0,1]$. We clearly have $p_t \in \mathcal{M}_n$, $p_0(z) = p(z)$ and $p_1(z) = q(z)$. Besides, $\|p_t - p\| = t\|p-q\| = t\varepsilon \leq \varepsilon$ for all $t \in [0,1]$ so that $p_t \in N_\varepsilon(p)$. Let $\varphi : [0,1] \to \mathbf{R}$ be the function $t \mapsto a(p_t)$. By continuity of a on $\mathcal{M}_n$ (see Proposition 5), the function φ is continuous. As $a(p) = \varphi(0) < 0$ and $a(q) = \varphi(1) \geq 0$, there is $\bar{t} \in [0,1]$ such that $a(p_{\bar{t}}) = 0$. There exists $y \in \mathbf{R}$ such that $p_{\bar{t}}(iy) = 0$. As the pseudozero set $Z_\varepsilon(p)$ is compact and contains an $iy, y \in \mathbf{R}$, we can take the intersection between the vertical line through iy and the pseudozero set. Let iy' be a point on the boundary of this intersection. It verifies $h_{p,\varepsilon}(0, y') = 0$. This complete the proof. □

4 An algorithm to compute the stability radius

We still assume that polynomial p is stable. As polynomial z^n is stable, is clear that $0 \leq \beta(p) \leq \|p - z^n\|$. We propose to apply a bisection algorithm to compute $\beta(p)$. The real γ and δ are respectively a lower and an upper

Algorithm 1 Computation of stability radius by bisection

Require: a stable polynomial p and a tolerance τ
Ensure: a number α such that $|\alpha - \beta(p)| \leq \tau$

```
1: γ := 0,   δ := ‖p − z^n‖
2: while |γ − δ| > τ do
3:    ε := (γ+δ)/2
4:    if the equation h_{p,ε}(0, y) = 0 has a real solution then
5:       δ := ε
6:    else
7:       γ := ε
8:    end if
9: end while
10: return α = (γ+δ)/2
```

bound for $\beta(p)$. As a consequence, we always have $\gamma \leq \beta(p) \leq \delta$. Because of the condition in the loop, we get α such that $|\alpha - \beta(p)| \leq \tau$ at the end of the algorithm. The parameter τ is an arbitrary tolerance that measures the accuracy of the stability radius $\beta(p)$. Indeed, we have $|\alpha - \beta(p)| \leq |\delta - \gamma| \leq \tau$.

The difficult step of the algorithm is to test if the polynomial $P(y) = h_{p,\varepsilon}(0, y)$ has real roots. The polynomial P has complex coefficients. Let us define the polynomial $Q = P\overline{P}$, where $\overline{P}$ is the complex conjugate polynomial of P. We easily see that Q is a polynomial with real coefficients and that P has a real root if and only if Q has also a real root. We can apply the Euclid's algorithm to Q and Q'. Let $Q_0 = Q$ and $Q_1 = Q'$ and define $Q_{i+1} = -\operatorname{rem}(Q_{i-1}, Q_i)$. Let m be the smallest integer such that $Q_{m+1} = 0$. Let be $v_Q(-\infty)$ the number of sign changes in the leading coefficients of $Q_0(-X), \ldots, Q_m(-X)$ and let $v_Q(+\infty)$ be the number of sign changes in the leading coefficients of $Q_0(X), \ldots, Q_m(X)$. So we have defined what is called a Sturm sequence and Q has exactly $v_Q(-\infty) - v_Q(+\infty)$ distinct real roots. In particular $h_\varepsilon(0, y)$ has a real root if and only if $v_Q(-\infty) \neq v_Q(+\infty)$. Let us remark that using Sturm sequences suffices to answer line 4 of Algorithm 1 without having to compute all the roots of $h_\varepsilon(0, y)$.

5 Numerical simulations

Algorithm 1 is implemented using the MAPLE software. This choice is motivated by the fact that we need some formal manipulations of polynomials.

For drawing pseudozero set, we used the MATLAB software to implement following Algorithm 2.

Algorithm 2 Computation of ε-pseudozero set

Require: polynomial p and precision ε

Ensure: pseudozero set layout in the complex plane

1: We grid a square containing the whole roots of p with the MATLAB command `meshgrid`.

2: We compute $g(z)$ for the whole points z on the grid.

3: We draw the level line $|g(z)| = \varepsilon$ with the MATLAB command `contour`.

A first natural example is $p(z) = z + 1$. Of course, the nearest unstable polynomial of p is $q(z) = z$ and then $\beta(p) = 1$. Algorithm 1 yields $\beta = 0.999996$ with a tolerance equals to 0.00001. We can draw the 0.999996-pseudozero set (see Fig 1) and we verify that the pseudozero set is included in the left half-plane and is tangent to the imaginary axis. This confirms the intuitive aspect of the algorithm.

Now we consider $p(z) = (z - 1)(z - 1/2) = z^2 + z + 1/2$. Algorithm 1 yields $\beta = 0.485868$ with the tolerance 0.00001. We can draw the 0.485868-pseudozero set (see Fig 2).

If we choose for example the polynomial $p(z) = z^3 + 4z^2 + 6z + 4$, we get $\beta = 2.610226$ with a tolerance 0.00001. We can draw the 2.610226-pseudozero set (see Fig 3).

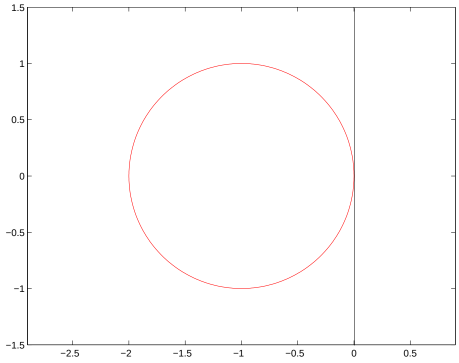

Figure 1: ε-pseudozero set for $p(z) = z + 1$ with $\varepsilon = 0.999996 \approx \beta(p)$

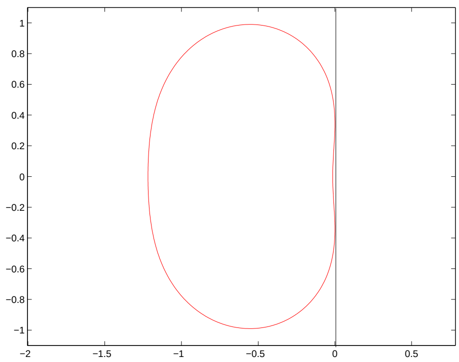

Figure 2: ε-pseudozero set for $p(z) = z^2 + z + 1/2$ with $\varepsilon = 0.485868 \approx \beta(p)$

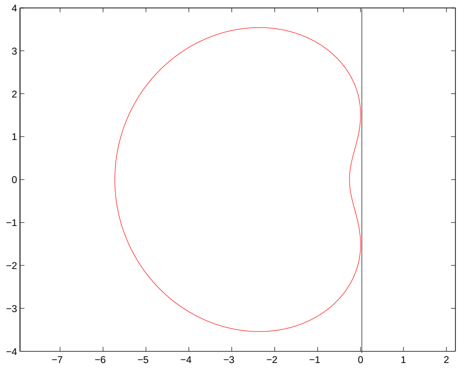

Figure 3: ε-pseudozero set for $p(z) = z^3 + 4z^2 + 6z + 4$ with $\varepsilon = 2.610226 \approx \beta(p)$

6 Conclusion

In this paper, we have shown the usefulness of the ε-pseudozero set introduced by Mosier [8]. The theory used in this concept is the basis of our algorithm. Indeed, we used the level contour function of the pseudozero set to derive a bisection algorithm that computes the stability radius of a polynomial. Moreover, the visualization of the ε-pseudozero set give us significant informations about the stability of a polynomial. These informations are both quantitative (the result of the bisection algorithm) and qualitative (the drawing). Although it seems that pseudozero set is not popular in applied mathematics, we hope that we have demonstrated the power of this numerical tool.

The case where the polynomial have real coefficients seems to be more difficult. Indeed, it seems there is no explicit formula to compute the ε-pseudozero set in the case of real perturbations.

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