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Mathematics in Computer Science

Pseudozero Set of Real Multivariate Polynomials

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Abstract. The pseudozero set of a system P of polynomials in n variables is the subset of $\mathbf{C}^n$ consisting of the union of the zeros of all polynomial systems Q that are near to P in a suitable sense. This concept arises naturally in Scientific Computing where data often have a limited accuracy. When the polynomials of the system are polynomials with complex coefficients, the pseudozero set has already been studied. In this paper, we focus on the case where the polynomials of the system have real coefficients and such that all the polynomials in all the perturbed polynomial systems have real coefficients as well. We provide an explicit definition to compute this pseudozero set. At last, we analyze different methods to visualize this set.

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1. Introduction and notation

1.1. Summary

Polynomials appear in almost all areas in scientific computing and engineering as it is shown in the Computer Algebra Handbook [8] and in [5]. Many of the applications need to solve equations involving polynomials and systems of polynomials, often in many variables. The relationships between industrial applications and polynomial systems solving were studied by the European Community Project FRISCO. The report may be found at `http://www.nag.co.uk/projects/FRISCO.html`. They gave a list of the major fields where polynomial systems are used: Computer Aided Design and Modeling, Mechanical Systems Design, Signal Processing and Filter Design, Civil Engineering, Robotics, Simulation. The wide range of use of polynomial systems needs to have fast and reliable methods to solve them. Roughly speaking, there are two general approaches: symbolic and numeric. The symbolic approach is based either on the theory of Gröbner basis or on the theory of resultants. For the numeric approach, it is the use of iterative methods like Newton's method or homotopy continuation methods. Recently, hybrid methods, combining both symbolic and numeric methods, began to appear (see the chapter called "Hybrid Methods" by Kaltofen *et al* in [8, p. 112–128]).

In practice, from situations arising in science or engineering, the data are known only to a limited accuracy. From a polynomial point of view, this only means that the coefficients of the polynomials are known only to within a certain tolerance. Then it is important to obtain informations about the variation of the zeros of the polynomial or of the polynomial system in the presence of uncertainty on the coefficients. Analytical sensitivity analysis introduces a condition number that bounds the magnitudes of the (first order) changes of the roots with respect to the coefficient perturbations. Numerous results in this direction are available, see for example Gautschi [7] or Wilkinson [29]. Representing coefficient uncertainty with intervals and computing with interval arithmetic yield over-sets that enclose (sometimes pessimistically) the perturbed roots. Continuous sensitivity analysis, introduced by Ostrowski [22], considers the uncertainty of the coefficients as a continuity problem. The most powerful tool of this last type of methods seems to be the *pseudozero set* of a polynomial we focus hereafter. Roughly speaking, this is the set of roots of polynomials that are near to a given polynomial.

The pseudozero set was first introduced by Mosier [21] in 1986. He studied this set considering perturbations bounded with the ∞-norm. Trefethen and Toh [28] studied pseudozero set for perturbations bounded with the 2-norm. They also compared the pseudozero set of a given polynomial with the pseudospectra of the associated companion matrix. These results are summarized in Chatelin and Frayssé's book on finite precision [3]. More recently, Zhang [30] compared pseudozero set with respect to the choice of the polynomial basis (power, Taylor, Chebyshev, Bernstein). At last, recently, Stetter gave a general framework for working with inexact polynomials in his book [27] (based on previous papers [24–26]). The notion of root sets was introduced by Hinrichsen and Kelb [14]. It is a particular case of the spectral value sets of the companion matrix using structured perturbations. It corresponds exactly to the notion of pseudozero set but from a different viewpoint. Such a set was studied in particular by Hinrichsen and Kelb [14], Karow [19] and Hinrichsen and Pritchard [16].

Nevertheless, few applications of pseudozero set have been given in these previous publications, except when Bini and Fiorentino provided a multiprecision algorithm to compute polynomial root using pseudozero set [1]. Indeed, they need to know if an approximate root is a root of a nearby polynomial. Pseudozero set is the natural way to answer this question. More recently, Graillat and Langlois [9–12] gave some applications of pseudozero set in Computer Algebra and in Control Theory. They provide in these articles an algorithm to test the approximate primality of two univariate polynomials (see also [2]). They also propose an algorithm to compute the stability radius of a univariate polynomial.

The major part of the papers cited above consider only the univariate case. The multivariate case seems to have received few attention. It has only been studied by Stetter in [25, 27], by Hoffman, Madden and Zhang in [17] and Corless, Kai and Watt in [4]. Furthermore, the multivariate case has only been dealt with polynomials with complex coefficients. In this paper, we consider systems where polynomials have real coefficients and such that all the polynomials in all the perturbed polynomial systems have real coefficients as well. We provide a simple criterion to compute the pseudozero set and study different methods to visualize it.

The rest of the paper is organized as follows. In the rest of this section, we introduce some notations and well-known results on basic linear algebra and computer algebra. In Section 2, we recall results on complex pseudozero set. In Section 3, we study real pseudozero set and establish a computable criterion for this pseudozero set. In Section 4, we present different methods to visualize the pseudozero set.

1.2. Notation

We recall the notations used in Stetter [27]. A *monomial* in the n variables $z_1, \ldots, z_n$ is the power product

$$z^j := z_1^{j_1} \cdots z_n^{j_n}, \quad \text{with} \quad j = (j_1, \ldots, j_n) \in \mathbf{N}^n;$$

j is the *exponent* and $|j| := \sum_{\sigma=1}^n j_\sigma$ the *degree* of the monomial z^j.

Definition 1.1. A *complex* (*real*) *polynomial* in n variables is a finite linear combination of monomials in n variables with coefficients from $\mathbf{C}$ (from $\mathbf{R}$),

$$p(z) = p(z_1, \ldots, z_n) = \sum_{(j_1,\ldots,j_n)\in J}^{n} a_{j_1\cdots j_n} z_1^{j_1} \cdots z_n^{j_n} = \sum_{j\in J} a_j z^j .$$

The set $J \subset \mathbf{N}^n$ which contains the exponents of those monomials which are present in the polynomial p is called the *support* of p. The total *degree* of p is defined to be the number $\deg(p) := \max_{j\in J} |j|$. The set of all complex (real) polynomials in n variables will be denoted by $\mathcal{P}^n(\mathbf{C})$ (by $\mathcal{P}^n(\mathbf{R})$). When the coefficient domain is evident or is not important, the notation $\mathcal{P}^n$ will be used. The notation $\mathcal{P}^n_d \subset \mathcal{P}^n$ stands for the set of polynomials in n variables of total degree $\leq d$. As we will often manipulate polynomials with linear operations, we will widely employ the notations of linear algebra. We will generally collect the coefficients of a polynomial into a vector $a = (\ldots, a_j, \ldots, j \in J)^T$ and its monomials into a vector $\mathbf{z} = (\ldots, z^j, \ldots, j \in J)^T$.

Let $p = \sum_{j\in J} a_j z^j \in \mathcal{P}^n(\mathbf{K})$ with $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ be a polynomial in n variables and J be its support. We denote by $|J|$ the number of elements of J. If $|J| = M$ and let $\|\cdot\|$ be a norm on $\mathbf{K}^M$, we denote by $\|p\|$ the norm of the vector $p = (\ldots, a_j, \ldots, j \in J)$, namely,

$$\|p\| := \|(\ldots, a_j, \ldots, j \in J)^T\| .$$

Given such an $\varepsilon > 0$, the ε-neighborhood $N_\varepsilon(p)$ of the polynomial $p \in \mathcal{P}^n(\mathbf{K})$ is the set of all polynomials of $\mathcal{P}^n(\mathbf{K})$, close enough to p, that is to say, the set of polynomials $\tilde{p} = \sum_{j\in\tilde{J}} \tilde{a}_j z^j \in \mathcal{P}^n(\mathbf{K})$ with support $\tilde{J} \subset J$ and $\|\tilde{p} - p\| \leq \varepsilon$.

Table 1. Dual norms for most common norms on $\mathbf{K}^N$.

Norms	Dual norms
$\|x\|_1 := \sum_j \|x_j\|$	$\|x\|_1^* = \max_j \|x_j\| = \|x\|_\infty$
$\|x\|_2 := (\sum_j \|x_j\|^2)^{1/2}$	$\|x\|_2^* = (\sum_j \|x_j\|^2)^{1/2} = \|x\|_2$
$\|x\|_\infty := \max_j \|x_j\|$	$\|x\|_\infty^* = \sum_j \|x_j\| = \|x\|_1$

Given a norm $\|\cdot\|$ on $\mathbf{K}^N$ with $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, we define its *dual norm* (denoted by $\|\cdot\|_*$) by

$$\|x\|_* := \sup_{y \neq 0} \frac{|y^T x|}{\|y\|} = \sup_{\|y\|=1} |y^T x| \,.$$

Table 1 represents the most common norms on $\mathbf{K}^N$ and their respective dual norms. Given a vector $x \in \mathbf{K}^N$, there exists a vector $y \in \mathbf{K}^N$ with $\|y\| = 1$ satisfying $x^T y = \|x\|_*$ (see [13, p. 107] or [18, p. 278]). The vector y is called the *dual vector* of x.

Definition 1.2. A value $z \in \mathbf{K}^n$ is a *ε-pseudozero* of a polynomial $p \in \mathcal{P}^n$ if it is a zero of some polynomial $\tilde{p}$ in $N_\varepsilon(p)$.

Definition 1.3. The *ε-pseudozero set* of a polynomial $p \in \mathcal{P}^n$ (denoted by $Z_\varepsilon(p)$) is the set of all the ε-pseudozeros,

$$Z_\varepsilon(p) := \left\{ z \in \mathbf{K}^n : \exists \tilde{p} \in N_\varepsilon(p), \quad \tilde{p}(z) = 0 \right\} .$$

Three important issues arise from these definitions.

- For p with real coefficients a_j, it must be specified whether $N_\varepsilon(p)$ is restricted to real polynomials or not. Indeed, it seems natural for a real polynomial to be perturbed by real polynomials.
- One may only be interested in real or complex pseudozero set.
- The pseudozero set $Z_\varepsilon(p)$ cannot be computed directly because it is the union of the zeros of an infinite number of polynomials.

We can extend those definitions to a *system* of polynomials

$$P = \{p_1, \ldots, p_k\} \,, \quad k \in \mathbf{N} \,.$$

We will often consider this system as a vectors of polynomials

$$P(z) = \begin{pmatrix} p_1(z) \\ \vdots \\ p_k(z) \end{pmatrix} .$$

Given an $\varepsilon > 0$ and a *system* of polynomials $P = \{p_1, \ldots, p_k\}$, $k \in \mathbf{N}$, the ε-neighborhood $N_\varepsilon(P)$ is the set of systems of polynomials $\tilde{P} = \{\tilde{p_1}, \ldots, \tilde{p_k}\}$ close enough to P, that is to say with $\tilde{p_j} \in N_\varepsilon(p_j)$ for $j = 1, \ldots, k$.

Definition 1.4. A value $z \in \mathbf{K}^n$ is a *ε-pseudozero* of a polynomial system P if it is a zero of a system of polynomials $\tilde{P}$ in $N_\varepsilon(P)$.

Definition 1.5. The *ε-pseudozero set* of a system of polynomials P (denoted by $Z_\varepsilon(P)$) is the set of all the ε-pseudozeros,

$$Z_\varepsilon(P) := \left\{ z \in \mathbf{K}^n : \exists \tilde{P} \in N_\varepsilon(P), \quad \tilde{P}(z) = 0 \right\}.$$

2. Pseudozero set of complex multivariate polynomials

Theorem 2.1 below provides a computable counterpart of the pseudozero set.

Theorem 2.1 (Stetter [27]). *The complex ε-pseudozero set of $p = \sum_{j\in J} a_j z^j \in \mathcal{P}^n(\mathbf{C})$ verifies*

$$Z_\varepsilon(p) = \left\{ z \in \mathbf{C}^n : g(z) := \frac{|p(z)|}{\|\mathbf{z}\|_*} \leq \varepsilon \right\},$$

where $\mathbf{z} := (\dots, |z|^j, \dots, j \in J)^T$.

For completeness of the paper, we recall the proof.

Proof. If $z \in Z_\varepsilon(p)$ then there exists $\tilde{p} \in \mathcal{P}^n$ such that $\tilde{p}(z) = 0$ and $\|p - \tilde{p}\| \leq \varepsilon$. From the generalized Hölder's inequality $|x^T y| \leq \|x\| \|y\|_*$, we get

$$|p(z)| = |p(z) - \tilde{p}(z)| = \Big| \sum_{j\in J} (p_j - \tilde{p}_j) z^j \Big| \leq \|p - \tilde{p}\| \|\mathbf{z}\|_*.$$

It follows that $|p(z)| \leq \varepsilon \|\mathbf{z}\|_*$.

Conversely, let $u \in \mathbf{C}$ be such that $|p(u)| \leq \varepsilon \|\mathbf{u}\|$ where $\mathbf{u} := (\dots, |u|^j, \dots, j \in J)$. The dual vector d of $\mathbf{u}$ verifies $d^T \mathbf{u} = \|\mathbf{u}\|_*$ and $\|d\| = 1$. Let us introduce the polynomials r and p_u defined by

$$r(z) = \sum_{k=0}^{n} r_k z^k \quad \text{with} \quad r_k = d_k,$$

$$p_u(z) = p(z) - \frac{p(u)}{r(u)} r(z).$$

This polynomial p_u is (with respect to the norm $\|\cdot\|$) the nearest polynomial of p with u as a root.

It is clear that $r(u) = d^T \mathbf{u} = \|\mathbf{u}\|_*$. So we have

$$\|p - p_u\| = \frac{|p(u)|}{|r(u)|} \|r\| \leq \varepsilon \|d\|.$$

As $\|d\| = 1$, we get

$$\|p - p_u\| \leq \varepsilon.$$

And since $p_u(u) = 0$, u belongs to $Z_\varepsilon(p)$. □

This theorem can be immediately extended to systems of polynomials.

Corollary 2.2 (Stetter [25]). *The complex ε-pseudozero set of $P = \{p_1, \ldots, p_k\}$, $k \in \mathbf{N}$ verifies*

$$Z_\varepsilon(P) = \left\{ z \in \mathbf{C}^n : \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \le \varepsilon \quad for \quad l = 1, \ldots, k \right\} ,$$

where $\mathbf{z_l} := (\ldots, |z|^j, \ldots, j \in J_l)^T$.

For the next theorem, we will restrict our attention to situations where P as well as all the systems in $N_\varepsilon(P)$ are 0-dimensional, that is to say if the solutions of the system are non-empty and finite.

Theorem 2.3 (Stetter [25]). *Under the above assumptions, each system $\tilde{P} \in N_\varepsilon(P)$ has the same number of zeros (counting multiplicities) in a fixed pseudozero set connected component of $Z_\varepsilon(P)$.*

Proof. We can copy the proof of [25, Thm. 3.5]. Because of the assumed uniform 0-dimensionality in $N_\varepsilon(P)$, the Jacobian $\tilde{P'}(z)$ can only be singular at a finite number of isolated points for each $\tilde{P} \in N_\varepsilon(P)$. At all other points $z \in \mathbf{C}^n$, $\tilde{P'}(z)$ is regular and, by the inverse function theorem, a full neighborhood of $\tilde{P}$ is mapped differentiably onto a full neighborhood of x. Thus, generally, a zero of

$$P_t(z) := (1-t)\tilde{P^*}(z) + t\tilde{P}(z) , \quad t \in [0, 1] ,$$

moves smoothly as a function of t because, at some $\bar{t} \in [0, 1]$, a small increment of t in P_t may be interpreted as a small perturbation of $P_{\bar{t}}$ which is a $\tilde{P} \in N_\varepsilon(P)$. If a zero $z(t)$ of P_t coincides with a singularity of P'_t on its way from $z(0)$ to $z(1)$, we can either locally replace the linear homotopy by a different one which guides $z(t)$ around the isolated singularity, or we can refer to the analysis of perturbations of polynomial systems at a multiple zero presented in [23]: no zeros can be gained or lost if the path of several $z(t)$ passes through a common multiple zero $z(\bar{t})$. □

3. Pseudozero set of real multivariate polynomials

3.1. Complex pseudozero set of real multivariate polynomials

A real ε-neighborhood of p is the set of all polynomials of $\mathcal{P}^n(\mathbf{R})$, close enough to p, that is to say,

$$N_\varepsilon^R(p) = \left\{ \tilde{p} \in \mathcal{P}^n(\mathbf{R}) : \|p - \tilde{p}\| \le \varepsilon \right\} .$$

Then the real ε-pseudozero set of p is defined to include all the zeros of the real ε-neighborhood of p. A definition of this set is

$$Z_\varepsilon^R(p) = \left\{ z \in \mathbf{C}^n : \tilde{p}(z) = 0 \quad \text{for} \quad \tilde{p} \in N_\varepsilon^R(p) \right\} .$$

For $\varepsilon = 0$, the pseudozero set $Z_0^R(p)$ is the set of the roots of p we denote $Z(p)$.

Following Theorem 3.1 provides a computable counterpart of this definition. It is based on arguments developed by Hinrichsen and Kelb in [14]. We define for $x, y \in \mathbf{R}^N$,

$$d(x, \mathbf{R}y) = \inf_{\alpha \in \mathbf{R}} \|x - \alpha y\|_* ,$$

the distance of a point $x \in \mathbf{R}^N$ from the linear subspace $\mathbf{R}y = \{\alpha y, \alpha \in \mathbf{R}\}$.

Theorem 3.1. *The real ε-pseudozero set of $p = \sum_{j\in J} a_j z^j \in \mathcal{P}^n(\mathbf{R})$ verifies*

$$Z_\varepsilon^R(p) = Z(p) \cup \left\{ z \in \mathbf{C}^n \backslash Z(p) : h(z) := d\big(G_R(z)\,, \mathbf{R}G_I(z)\big) \geq \frac{1}{\varepsilon} \right\} ,$$

where $G_R(z)$ and $G_I(z)$ are the real and imaginary parts of

$$G(z) = \frac{1}{p(z)}(\dots, z^j, \dots, j \in J)^T , \quad z \in \mathbf{C}^n \backslash Z(p)\,.$$

Proof. Let $z \in Z_\varepsilon^R(p)$. If $p(z) = 0$ then $z \in Z(p)$ else there exists $q \in N_\varepsilon^R(p)$ such that $q(z) = 0$. In this case, we have $p(z) = p(z) - q(z) = (p-q)^T \underline{z}$, where $\underline{z} = (\dots, z^j, \dots, j \in J)^T$. It follows that $1 = (p-q)^T G(z)$. Hence we have $1 = (p-q)^T G_R(z) + i(p-q)^T G_I(z)$ and so

$$\begin{cases} (p-q)^T G_R(z) = 1\,, \\ (p-q)^T G_I(z) = 0\,. \end{cases}$$

As a consequence, we have $\|p-q\|\|G_R(z) - \alpha G_I(z)\|_* \geq 1$, for all $\alpha \in \mathbf{R}$. We conclude that

$$d\big(G_R(z), \mathbf{R}G_I(z)\big) \geq \frac{1}{\|p-q\|} \geq \frac{1}{\varepsilon}\,.$$

Conversely, let $z \in Z(p) \cup \big\{ z \in \mathbf{C}^n \backslash Z(p) : d(G_R(z), \mathbf{R}G_I(z)) \geq \frac{1}{\varepsilon} \big\}$. If z belongs to $Z(p)$ then it belongs to $Z_\varepsilon^R(p)$. Otherwise z satisfies $d(G_R(z), \mathbf{R}G_I(z)) \geq 1/\varepsilon$. From a duality theorem (see [20, p. 119]), there exists a vector $u \in \mathbf{R}^N$ with $\|u\| = 1$ satisfying

$$u^T G_R(z) = d\big(G_R(z), \mathbf{R}G_I(z)\big) \quad \text{and} \quad u^T G_I(z) = 0\,.$$

Let us consider the real polynomial

$$q = p - \frac{u}{d\big(G_R(z), \mathbf{R}G_I(z)\big)}\,.$$

We have

$$q(z) = p(z) - \frac{u^T \underline{z}}{d\big(G_R(z), \mathbf{R}G_I(z)\big)} = p(z) - \frac{p(z) u^T G(z)}{d\big(G_R(z), \mathbf{R}G_I(z)\big)} = 0\,.$$

Furthermore we have $\|q-p\| = 1/d(G_R(z), \mathbf{R}G_I(z))$, so that $\|p-q\| \leq \varepsilon$. □

To compute the real ε-pseudozero set $Z_\varepsilon^R(p)$, we only have to evaluate the distance $d\big(G_R(z), \mathbf{R}G_I(z)\big)$. This quantity can be calculated easily for the 2-norm. Let us now denote the 2-norm $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$ the corresponding inner product. In this case, we have

$$d(x, \mathbf{R}y) = \begin{cases} \sqrt{\|x\|_2^2 - \frac{\langle x,y\rangle^2}{\|y\|_2^2}} & \text{if } y \neq 0\,, \\ \|x\|_2 & \text{if } y = 0\,. \end{cases}$$

For the ∞-norm, it is shown in [19, Prop. 7.7.2] that

$$d(x, \mathbf{R}y) = \begin{cases} \min_{\substack{i=0:n \\ y_i \neq 0}} \|x - (x_i/y_i)y\|_1 & \text{if } y \neq 0\,, \\ \|x\|_1 & \text{if } y = 0\,. \end{cases}$$

For the other p-norm with $p \neq 2, \infty$, as far as the author knows, there is no easy known computable formula to calculate $d(x, \mathbf{R}y)$.

This theorem can be immediately extended to systems of polynomials.

Corollary 3.2. *The real ε-pseudozero set of $P = \{p_1, \dots, p_k\}$, $k \in \mathbf{N}$ verifies*

$$Z_\varepsilon^R(P) = \bigcap_{l=1}^{k} \left(Z(p_l) \cup \left\{ z \in \mathbf{C}^n \backslash Z(p_l) : h_l(z) := d\big(G_R^l(z), \mathbf{R}G_I^l(z)\big) \geq \frac{1}{\varepsilon} \right\} \right),$$

where $G_R^l(z)$ and $G_I^l(z)$ are the real and imaginary parts of

$$G^l(z) = \frac{1}{p_l(z)}(\dots, z^j, \dots, j \in J_l)^T\,, \quad z \in \mathbf{C}^n \backslash Z(p_l)\,.$$

As we have seen before, the real pseudozero set is closely related to the function d. This function can have a discontinuous behavior. It is the subject of the following lemma.

Lemma 3.3 (Hinrichsen and Kelb [14]). *The function*

$$d : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}_+\,, \quad (x, y) \mapsto d(x, \mathbf{R}y)$$

is continuous at all pairs (x, y) with $y \neq 0$ or $x = 0$ and discontinuous at all pairs $(x, 0) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, $x \neq 0$.

This lemma states that a discontinuity problem arises when vector y vanishes. In our case, the discontinuity arises when $G_I(z) = 0$ where $G_I(z)$ is the imaginary part of

$$G(z) = \frac{1}{p(z)}(1, z, \dots, z^n)^T\,.$$

It follows that G_I vanishes for $z \in \mathbf{R}$, that is along the real axis. This explains why the `contour` and `meshc` functions of MATLAB may give some bad results along the real axis. Of course, if none of the zeros of the polynomial is real, the real pseudozero set is correct because we do not evaluate the function G on the real axis.

3.2. Real pseudozero set of real multivariate polynomials

In the previous subsection, we were interested in the complex zeros of a real polynomial system. Sometimes, we can be interested only in the real zeros of a system. That is to say, given a polynomial $p \in \mathcal{P}^n(\mathbf{R})$, we are interested in $Z_\varepsilon^R(p) \cap \mathbf{R}^n$. The following result gives a formula to compute this set.

Theorem 3.4. *The intersection between the complex ε-pseudozero set of $p = \sum_{j\in J} a_j z^j \in \mathcal{P}^n(\mathbf{C})$ and $\mathbf{R}^n$ verifies*

$$Z_\varepsilon^R(p) \cap \mathbf{R}^n = \left\{ z \in \mathbf{R}^n : g(z) := \frac{|p(z)|}{\|\mathbf{z}\|_*} \leq \varepsilon \right\} ,$$

where $\mathbf{z} := (\ldots, |z|^j, \ldots, j \in J)^T$.

Proof. If $z \in Z_\varepsilon^R(p) \cap \mathbf{R}^n$ then there exists $\tilde{p} \in \mathcal{P}^n(\mathbf{R})$ such that $\tilde{p}(z) = 0$ and $\|p - \tilde{p}\| \leq \varepsilon$. From the generalized Hölder's inequality $|x^T y| \leq \|x\|\|y\|_*$, we get

$$|p(z)| = |p(z) - \tilde{p}(z)| = \left| \sum_{j\in J} (p_j - \tilde{p}_j) z^j \right| \leq \|p - \tilde{p}\| \|\mathbf{z}\|_* .$$

It follows that $|p(z)| \leq \varepsilon \|\mathbf{z}\|_*$.

Conversely, let $u \in \mathbf{R}$ be such that $|p(u)| \leq \varepsilon \|\mathbf{u}\|$ where $\mathbf{u} := (\ldots, |u|^j, \ldots, j \in J)$. The dual vector $d \in \mathbf{R}^N$ of $\mathbf{u}$ verifies $d^T \mathbf{u} = \|\mathbf{u}\|_*$ and $\|d\| = 1$. Let us introduce the polynomials r and p_u defined by

$$r(z) = \sum_{k=0}^{n} r_k z^k \quad \text{with} \quad r_k = d_k ,$$

$$p_u(z) = p(z) - \frac{p(u)}{r(u)} r(z) .$$

This polynomial p_u is (with respect to the norm $\|\cdot\|$) the nearest polynomial of p with u as a root. It is clear that $r(u) = d^T \mathbf{u} = \|\mathbf{u}\|_*$. So we have

$$\|p - p_u\| = \frac{|p(u)|}{|r(u)|} \|r\| \leq \varepsilon \|d\| .$$

As $\|d\| = 1$, we get

$$\|p - p_u\| \leq \varepsilon .$$

And since $p_u(u) = 0$, u belongs to $Z_\varepsilon^R(p) \cap \mathbf{R}^n$. □

This theorem can be immediately extended to systems of polynomials.

Corollary 3.5. *The intersection between the complex ε-pseudozero set of $P = \{p_1, \ldots, p_k\}$, $k \in \mathbf{N}$ and $\mathbf{R}^n$ verifies*

$$Z_\varepsilon^R(P) \cap \mathbf{R}^n = \left\{ z \in \mathbf{R}^n : \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \leq \varepsilon \ \ \text{for} \ \ l = 1, \ldots, k \right\} ,$$

where $\mathbf{z_l} := (\ldots, |z|^j, \ldots, j \in J_l)^T$.

4. Visualization of pseudozero sets

The descriptions of $Z_\varepsilon(P)$ and $Z_\varepsilon^R(P)$ given by Theorem 2.1 and Theorem 3.1 (and by Corollary 2.2 and Corollary 3.2) enable us to compute, plot and visualize pseudozero set of multivariate polynomials. The pseudozero set is a subset of $\mathbf{C}^n$ which can only be seen by its projections on low dimensional spaces that is often $\mathbf{C}$.

We have written a MATLAB program to compute and visualize these projections (see Appendix A). This program requires the Symbolic Math Toolbox (and the Extended Symbolic Math Toolbox) which is the MATLAB gateway to the kernel of MAPLE.

For a given $v \in \mathbf{C}^n$, let $Z_\varepsilon(P, j, v)$ be the projection of $Z_\varepsilon(P)$ onto the z_j-space around v. Then, it follows that for $P = \{p_1, \dots, p_k\}$,

$$Z_\varepsilon(P, j, v) = \left\{ z \in \mathbf{C}^n : z_i = v_i \quad \text{for} \quad i \neq j, \quad \text{and} \quad \max_{l=1,\dots,k} \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \leq \varepsilon \right\},$$

where $\mathbf{z_l} := (\dots, |z|^j, \dots, j \in J_l)^T$. One way for visualizing $Z_\varepsilon(P, j, v)$ is to plot the values of the projection of

$$\mathrm{ps}(z) := \log_{10} \left(\max_{l=1,\dots,k} \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \right)$$

over a set of grid points around v in z_j-space. In the same way, we define for a given $v \in \mathbf{C}^n$, $Z_\varepsilon^R(P, j, v)$ by the projection of $Z_\varepsilon^R(P)$ onto the z_j-space around v. Then, it follows that for $P = \{p_1, \dots, p_k\}$,

$$Z_\varepsilon^R(P, j, v) = \left\{ z \in \mathbf{C}^n : z_i = v_i \quad \text{for} \quad i \neq j, \text{ and } \max_{l=1,\dots,k} d\big(G_R^l(z), \mathbf{R} G_I^l(z)\big)^{-1} \leq \varepsilon \right\},$$

where $G_R^l(z)$ and $G_I^l(z)$ are the real and imaginary parts of

$$G^l(z) = \frac{1}{p_l(z)} (\dots, z^j, \dots, j \in J_l)^T, \quad z \in \mathbf{C}^n \backslash Z(p).$$

One way for visualizing $Z_\varepsilon^R(P, j, v)$ is still to plot the values of the projection of

$$\mathrm{ps}^R(z) := \log_{10} \left(\max_{l=1,\dots,k} d\big(G_R^l(z), \mathbf{R} G_I^l(z)\big)^{-1} \right)$$

over a set of grid points around v in z_j-space. We examine the following system from [17] (see Figure 1) using the 2-norm: two unit balls intersection at $(2, 2)$,

$$P_1 = \begin{cases} p_1 = (z_1 - 1)^2 + (z_2 - 2)^2 - 1, \\ p_2 = (z_1 - 3)^2 + (z_2 - 2)^2 - 1. \end{cases}$$

We might only be interested in the real zeros of a polynomial systems. In this case, we can only draw $\mathbf{R}^n \cap Z_\varepsilon^R(P)$. This is what is done with the following example from [4] in Figure 2 still with the 2-norm,

$$P_2 = \begin{cases} p_1 = z_1^2 + z_2^2 - 1, \\ p_2 = 25 z_1 z_2 - 12. \end{cases}$$

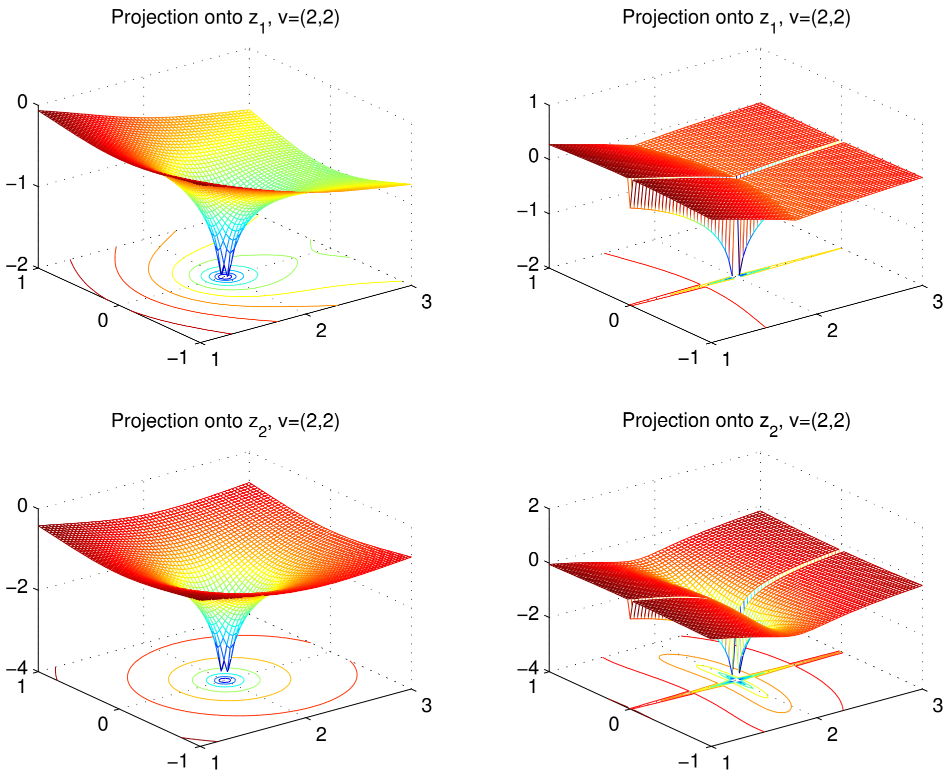

Figure 1. Projections of the complex pseudozero set (on the left) and the real pseudozero set (on the right) of P_1.

In this Figure, we have computed the function

$$g(x, y) = \max_{l=1,2} \frac{p_l(x, y)}{\|\mathbf{z_l}\|_*},$$

where $\mathbf{z_l} := (\dots, |x + iy|^j, \dots, j \in J_l)^T$.

Several issues appear when one wants to draw the real or complex pseudozero set. First, one has to choose a discretization that separates the roots. This is often a difficult task. For drawing the real pseudozero set, one needs to deal with function d that is discontinuous on the real axis.

The cost of our algorithms strongly depends on the number of nodes of the grid which can be very important. Nevetheless, we are not interested in providing cheap algorithms. We just want to provide tools that enable us to make a qualitative analysis of a polynomial.

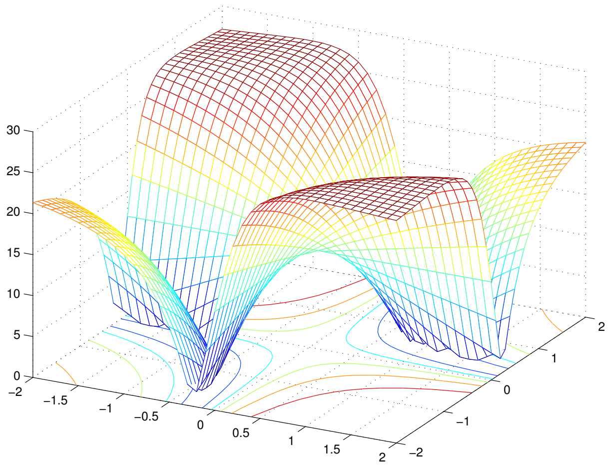

FIGURE 2. Projection of the real pseudozero set of P_2.

5. Conclusion

Approximate polynomials are unavoidable in numerous application fields and in finite precision environment. Plotting pseudozero set can give qualitative and sometimes quantitative interesting informations about the behavior of these approximate polynomials. We have shown that pseudozero set offers a powerful tool. They can be easily plotted using popular software as MATLAB. We hope that pseudozero set will be used as much as pseudospectra.

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Appendix A. MATLAB code

```
function [] = pseudo(polys,indets,proj,coord,xaxis,yaxis)
% polys   : system of polynomials
% indets  : variables
% proj    : variable where we project
% coord   : coordinate of the point near we project
% xaxis   : coordinate for the x-axis
% yaxis   : coordinate for the y-axis
% example :
% pseudo({'(x-1)^2+(y-2)^2-1','(x-3)^2+(y-2)^2-1'}, ...
```

```
%         {'x','y'},'x',[2 2],1:0.02:3,-1:0.02:1)

% load of a maple function that
% give the list of the monomial
% of a polynomial
procread('monomial.maple');

% number of variable in the system
nbindets = length(indets);

% put the variables as symbolic variables
for k = 1:nbindets
   syms(indets{k});
end

% number of polynomials in the system
nbpoly = length(polys);

monomials = {};
for k=1:nbpoly
   monomials{k} = maple('monomial',polys{k});
end

% substitute a value to variables which do not change
ind = 0; % index of the variable that moves
for k = 1:nbindets
   if (indets{k} ~= proj);
      for j=1:nbpoly
         polys{j} = simplify(subs(polys{k},indets{k},coord(k)));
         dual{j} = simplify(subs(monomials{j},indets{k},coord(k)));
      end
   else
      ind = k;
   end
end

x= xaxis;
y= yaxis;

% Define a grid
[X,Y] = meshgrid(x,y);

% size of the grid
[r,s] = size(X);
```

```
% Transform (x,y) of the grid in complex numbers as z=x+iy
Z = X + i.*Y;

for l=1:r
   for j=1:s
      tab = [];
      for k=1:nbpoly
         % compute the function that check the pseudozero set
         num = subs(polys{k},indets{ind},Z(l,j));
         denum = norm(subs(dual{k},indets{ind},Z(l,j)),2);
         tab = [log10(abs(num)/abs(denum)) tab];
      end
      Res(l,j) = max(tab);
   end
end

% draw the result
meshc(x,y,Res);
```

In the previous program, we use the following MAPLE function.

```
monomial := proc(poly)

local listmono,mono,nbmono,k,p;

listmono := [op(expand(poly))];
nbmono := nops(listmono);

for k from 1 to nbmono do
   mono := listmono[k];
   mono := simplify(abs(mono/coeffs(mono)));
   listmono[k] := mono;
od;
return(listmono);
end;
```

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